Volatility and its Measurements: The Design of a Volatility Index and the Execution of its Historical Time Series at the DEUTSCHE BÖRSE AG

Diplomarbeit an der Fachhochschule Würzburg-Schweinfurt
Fachbereich Betriebswirtschaft
Zur Erlangung des akademischen Grades
Diplom-Betriebswirt (FH)

vorgelegt von Lyndon Lyons
bei Prof. Dr. Notger Carl
im Fach Bank-, Finanz- und Investitionswirtschaft
in WS 2004/2005

Table of Contents

Table of Equations:

List of Abbreviations

AG	Aktiengesellschaft
ARCH	Autoregressive conditional heteroskedasticity
ATM	At the money
CBOE	Chicago Board of Option Exchange
CET	Central European Time
DAX	Deutsche Aktien Index
EONIA	Euro-Overnight-Index-Average
EUR	Euros
EUREX	European Exchange
EURIBOR	Euro Interbank Offered Rate
EUROSTOXX 50	Eurozone blue chip index
GARCH	Generalized autoregressive conditional heteroskedasticity
M	Month
OTC	Over the counter
OTM	Out of the money
PDE	Partial Differential Equation
REX	Rentenindex
S&P 100	Standard and Poor 100 index
S&P 500	Standard and Poor 500 index
SDE	Stochastic Differential Equation
SMI	Swiss Market Index
VDAX	Volatility index on the DAX options
VIX	CBOE's volatility index on the S&P 100 options
VOLAX	Futures contract on the VDAX
VSMI	Volatility index on the SMI options
VSTOXX	Volatility index on the EUROSTOXX 50 options
Y	Year

1. Introduction

The volatility index, sometimes called by financial professionals and academics as "the investor gauge of fear" has developed overtime to become one of the highlights of modern day financial markets. Due to the many financial mishaps during the last two decades such as LTCM (Long Term Capital Management), the Asian Crisis just to name a few and also the discovery of the volatility skew, many financial experts are seeing volatility risk as one of the prime and hidden risk factors on capital markets. This paper will mainly emphasize on the developments in measuring and estimating volatility with a concluding analysis of the historical time series of the new volatility indices at the Deutsche Boerse.

As a result of the volatility's increasing importance as a risk indicator and hedging instrument, many financial market operators and their institutional clients have pioneered and ventured out into developing methods of estimating and measuring volatility based on various well established academic models and eventually have even based their estimations on self-made models. Some established models have proven not to withstand the test of time and empirical data. The Black-Scholes Options Pricing model for instance, does not allocate for stochastic volatility (i.e. skewness). On the other hand, two models have gained importance over the years, namely the Stochastic Volatility Model and the GARCH (1,1). An insight into these three models will be carried out in this paper.

Two measurements which are widely used by financial and risk management practitioners to determine levels of volatility risk are the historical (realized) volatility, and the implied volatility. These two perspectives of volatility will be viewed with the emphasis being placed on the latter.

Two volatility trading strategies would be introduced, namely the straddle and trading in volatility and variance swaps. Then the old and new methodologies of calculating the volatility index at the Deutsche Börse AG will be discussed and the business case behind the concept of a volatility index will then be presented. Finally an analysis and interpretation of the calculated historical time series between years 1999 and 2004 of the new volatility indices will be done.

2. Volatility and its Measurements

An option is a financial contract which gives the right but not the obligation to buy (call) or to sell (put) a specific quantity of a specific underlying, at a specific price, on (European) or up to (American), a specified date. Such an option is called a plain vanilla option. An underlying of an option could be stocks, interest rate instruments, foreign currencies, futures or indices. Option buyers (long positions) usually pay an option premium (option price) to the option seller (short positions) when entering into the option contract. In return, the seller of the option agrees to meet any obligations that may occur as a result of entering the contract.

The options called exotics include Path-dependent options whereby its payoffs are dependent on the historical development of the underlying asset, such as the average price (Asian Option) or the maximum price (Lookup option) over some period of time. Then there are other options in which their payoffs are anchored on whether or not the underlying asset reaches specified levels during the contractual period. They are called Barrier options. Option traders are constantly faced with a dynamically altering volatility risk. While many speculate on the course volatility will take in the near future, some may tend to seek to hedge this risk. For instance Carr and Madan[1] suggested a strategy that combines the holding of static options, all the out-of-the money ones, and dynamically trading the underlying asset. Such a strategy is very costly and most of the time not convenient for most traders. That's why advances have been made to develop new products and strategies which allow investors and traders to hedge their portfolios of derivative assets as well as portfolio of basic assets against pure volatility exposure. Brenner and Galai[2] were one of the first researchers to suggest developing a volatility index back in 1989.

Then in 1993, Robert Whaley developed the first volatility index on S&P 100 options for the Chicago Board of Options Exchange (CBOE) which was then subsequently introduced in the same year. Called the VIX, it used the model described by Harvey and Whaley [1992][3] in their research article. One year afterwards in December 1994, the Deutsche Boerse started publishing its own volatility index on DAX options called

[1] Carr, P. and D. Madan, 1998 "Towards a Theory of Volatility Trading", Volatility: New Estimation Techniques for Pricing Derivatives, R. Jarrow editor, Risk Books, London, 417-427.
[2] Brenner, M. and D. Galai, 1989, „New Financial Instruments for Hedging Changes in Volatility", Financial Analyst Journal, July/August, 61-65
[3] Harvey, C.R. and R.E. Whaley, 1992, "Dividends and S&P 100 index option valuation", Journal of Futures Markets 12(2), 123-137

the VDAX on a daily basis. The Deutsche Boerse even went on further to introduce the first futures on volatility based on the VDAX called VOLAX in 1998.

To understand the concept behind a volatility index one must first understand the differences between the methods of volatility measurements and their forecasting abilities. Using the formula derived by Black and Scholes[4] to price options, one needs among other things, the parameter volatility. They derived a formula for plain vanilla options using the parameters listed below as input.

1. The current price of the underlying at time $t = S$
2. The strike price of the option = K
3. The time to expiration of the option = $T - t$
4. The risk free interest rate = r
5. The annualized volatility of the underlying (based on lognormal returns) = s

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$N(d) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{d} e^{-x^2/2} dx,$$

$$d_1 = \frac{\ln(S/K) + (r + s^2/2)(T-t)}{s\sqrt{T-t}}, \quad d_2 = d_1 - s\sqrt{T-t},$$

Equation 1: Black Scholes Option Pricing Model- Explicit Solution for a Call Price

Of all these parameters, only volatility is not observable in the market. As a result a large number of researches on estimating and forecasting volatility over the past decades have taken place. Given C, i.e. (the observable current market price of the underlying asset) one can equate the implied volatility using the Black and Scholes formula illustrated above. This is a typical method of estimating the volatility for a given underlying. Suppose a call option on the underlying is actively traded, then the option price is readily obtainable. So in equation (1) above, one calculates the (implied) volatility which would have been used within the formula to give the current market prices as the result. Such an implied volatility can then be used to price other options on that same underlying which are not frequently actively traded or for which prices are not normally available.

[4] Black, F. and M. Scholes, 1973, “The pricing of options and corporate liabilities”, Journal of Political Economy 81, 637-659.

The Black and Scholes model assumes constant volatility, however observed market prices for identical options with different strikes (exercise prices) and maturities show the opposite. Actual market observations conveyed skewness (smile) of volatility i.e. identical options with different strikes possessing different implied volatilities. More insight to the Black and Scholes formula will occur later on in this paper.
Volatility, standard deviation and risk are sometime used interchangeably by financial practitioners but in fact there are some conceptual differences. Poon and Granger[5] in there research article clarifies that in Finance, volatility is used to refer to standard deviation, s or variance, s^2 calculated from a set of observations. They further go on to state that the sample standard deviation in the field of Statistics is a distribution free parameter depicting the second moment characteristic of the sample data. When s is attached to a standard distribution, like that of the normal or the Student-t distributions, only then can the required probability density and cumulative probability density be analytically derived. As a scale parameter, s factorizes or reduces the size of the fluctuations generated by the Wiener process (which is assumed in the Black-Scholes model and other option pricing model) in a continuous time setting. The pricing dynamic of the pricing model is heavily dependent on the dynamic of the underlying stochastic process and whether or not the parameters are time varying. That's why Poon and Granger go on to point out that it is meaningless to use s as a risk measure unless it is attached to a distribution or a pricing dynamic. For example, in the Black and Scholes model a normal distribution $N(d)$ is assumed, as shown in equation (1).
Generally there are two methodologies for estimating volatility. As mentioned above, implied volatility reflects the volatility of the underlying asset given its market's option price. This volatility is forward looking. The second method is that of the historical or realized volatility. This is derived from recent historical data of annualized squared log returns of the option prices observed in the past on the options market. The main question in modern day research on volatility is to find out which one of the two measurements of volatility is better at forecasting true market volatility. Since there are several methods of calculating these two forms of volatility measurements, at this point a closer look at different methods of volatility measurement will be discussed below.

[5] Poon, S-H and C. Granger, 2002, "Forecasting volatility in financial markets: a review", 1-10

2.1 Historical (Realized) Volatility

The three methodologies which will be looked at in this section to estimate historical volatility are the most discussed in financial literature. The first is called the Close-Close Volatility Estimator which is also known as the "classical" estimator. Then there is the High-Low Volatility Estimator from Parkinson[6], which is considered by many to be far superior to the classical method because it incorporates the intraday high and low prices of the financial asset into its estimation of volatility. The third method of historical volatility estimation is the High-Low-Open-Close Volatility Estimator first put forward by Garman and Klass[7] [1980]. The latter two estimators are considered to be extreme-value estimators of volatility.

2.1.1 Close-Close Volatility Estimator

Before the estimators of historical volatility are introduced, the fundamental assumptions on which the estimation procedures are built upon will be introduced at this point. These assumptions are widely accepted today by financial faculties[8]. The random walk[9] has been used to describe the movement of stock prices for quite sometime now, even before Brownian motion. Even Black and Scholes[10] used the good approximation of a random walk in stock prices by implementing $\ln S$ in their Noble Prize winning option pricing formula. In his paper, Parkinson[11] utilizing some fundamentals of Statistical Physics compared the diffusion constant with that of the variance of stock price movement in the financial markets. He goes on to state, "*Suppose a point particle undergoes a one-dimensional, continuous random walk with a diffusion constant* D.

[6] Parkinson M., 1980, "The Extreme Value Method for Estimating the Variance of the Rate of Return", Journal of Business, 1980, Volume 53 (No. 1), 61-65.
[7] Garman M.B., M.J. Klass, 1980, "On the Estimation of Security Price Volatility from Historical Data", Journal of Business, 1980, Vol. 53 (No. 1), 67-78.
[8] See articles referred to in endnotes 6 and 7.
[9] Cootner, P., ed. 1964, "The Random Character of Stock Prices", Cambridge Mass., MIT Press.
[10] Black, F. and M. Scholes, 1973, "The pricing of options and corporate liabilities", Journal of Political Economy 81, 637-659.
[11] Parkinson M., 1980, "The Extreme Value Method for Estimating the Variance of the Rate of Return", Journal of Business, 1980, Volume 53 (No. 1), p. 62.

Then, the probability of finding the particle in the interval $(x, x+dx)$ *at time* t*, if it started at point* x_0 *at time* $t=0$*, is* $\frac{dx}{\sqrt{2pDt}}\exp\left[\frac{-(x-x_o)^2}{2Dt}\right]$*. By comparison with the normal distribution, we see that* D *is the variance of the displacement* $x-x_0$ *after a unit time interval. This suggests the traditional way to estimate D… Then, defining* d_i *= displacement during the* i *th interval,* $d_i = x(i)-x(i-1), i=1,2,...,n$*, we have*

$$D_x = \frac{1}{n-1}\sum_{i=1}^{n}(d_i-\overline{d})^2$$

Equation 2: Diffusion Constant

as an estimate for D*;*

$$\overline{d} = \frac{1}{n-1}\sum_{m=1}^{n} d_m = \text{mean displacement.”}$$

Using this approach the transformed (logarithmic) price, changes over any time interval in a normally distributed manner[12] with mean zero and variance proportional to the length of the interval and exhibits continuous sample paths. But it is not assumed that these paths may be observed everywhere. This is due to the restrictions that trades often occur only at discrete points in time and exchanges are normally closed during certain periods of time. Therefore having a series of stock prices $(S_1, S_2, ..., S_{n+1})$ which are quoted at equal intervals of unit of time; equaling $r_i = \ln(\frac{S_{i+1}}{S_i}), i=1,2,...$, $\overline{r}$ = mean rate of return is zero, annual number of trading days = 252 days and n = rate of return over i th time interval, then the annualized Close-Close Estimator s_{cc} is simply the classical definition of standard deviation which also happens to be the square root of the diffusion constant definition D (see equation (4) above).

$$s_{cc} = \sqrt{252.\frac{1}{n}.\sum_{i=1}^{n} r_i^2}$$

Equation 3: Close-Close Volatility Estimator

---

12 Garman M.B., M.J. Klass, 1980, “On the Estimation of Security Price Volatility from Historical Data”, Journal of Business, 1980, Vol. 53 (No. 1), 67-78.

This is the easiest method to estimate volatility. This formula, as shown above, only uses the market closing prices i.e. their logarithms to estimate the volatility. Garman and Klass[13] mentioned that advantages of the Close-Close estimator are it's simplicity of usage and its freedom from obvious sources of error and bias on the part of market activity. The most critical disadvantage of this estimator is its inadequate usage of readily available information such as opening, closing, high and low daily prices in its estimation. Such information could contribute to more efficiency in estimating volatility.

2.1.2 High-Low Volatility Estimator

Staying with the assumption made above Parkinson[14] introduced one of the first and widely accepted extreme value methods of estimating volatility. In his article he concluded that the diffusion constant of the underlying random walk of the stock price movements is the true variance of the rate of return of a common stock over a unit of time. He also proved in his article that the use of extreme values in estimating the diffusion constant provides a significantly better estimate. So he then recommended that estimates of variance of the rate of return should also make use of this extreme value method.

He goes on further to add that due to the fact daily, weekly and monthly highs (H) and lows (L) of prices of equities are readily available; it should be very easy to apply in practice. So using extreme values (i.e. minimum and maximum values) to estimate the diffusion constant, ceteris paribus, then let $(x_{\max} - x_{\min}) \leq l$ during time interval t. To ensure that the observed set $(l_1, l_2, ..., l_n)$ originates from a random walk of the kind mention above, the factor $\frac{1}{4\ln 2}$ is used[15]. Hence the extreme value estimate for the diffusion constant D is:

$$D_l = \frac{1}{4\ln 2} \cdot \frac{1}{n} \sum_{i=1}^{n} l_i^2$$

Equation 4: Extreme Value Diffusion Constant

[13] See footnote 12.
[14] Parkinson M., 1980, "The Extreme Value Method for Estimating the Variance of the Rate of Return", Journal of Business, 1980, Volume 53 (No. 1), 61-65.
[15] See calculation of random walk test factor in: Parkinson M., 1980, "The Extreme Value Method for Estimating the Variance of the Rate of Return", Journal of Business, 1980, Volume 53 (No. 1), 62-63.

Applying to the stock market let $l = \ln(\frac{H}{L})$ and the annualized High-Low Volatility Estimator (square root of the diffusion constant definition) s_{HL} can be calculated using:

$$s_{HL} = \sqrt{\frac{1}{4\ln 2} . \frac{252}{n} \sum_{i=1}^{n} \ln(\frac{H_i}{L_i})^2}$$

Equation 5: High-Low Volatility Estimator

These extremes values give more detail of the movements throughout the period, so such an estimator is much more efficient than the Close-Close estimator. A practical importance of this approach is the improved efficiency due to the fact that fewer observations are necessary in order to obtain the same statistical precision as the Close-Close volatility estimator.

2.1.3 High-Low-Open-Close Volatility Estimator

Building on the Parkinson's estimator, Garman and Klass[16] introduced in their article an volatility estimator which incorporated not only the high and low historical prices but also the open and closing historical indicators of stock price movements in estimating variance and hence volatility. Their assumptions were the same as mentioned in section 2.1.1 but extended to include the assumption that stock prices follow a geometric Brownian motion. The annualized High-Low-Open-Close volatility estimator s_{HLOC} from Garman and Klass is illustrated as

$$s_{HLOC} = \sqrt{\frac{252}{n} \sum_{i=1}^{n} \left[\frac{1}{2} . \left(\ln \frac{H_i}{L_i} \right)^2 - (2\ln 2 - 1) . \left(\ln \frac{C_i}{O_i} \right)^2 \right]}$$

Equation 6: High-Low -Open-Close Volatility Estimator

where by,

O = opening price of the period

C = closing price of the period

[16] Garman M.B., M.J. Klass, 1980, "On the Estimation of Security Price Volatility from Historical Data", Journal of Business, 1980, Vol. 53 (No. 1), 67-78.

The efficiency gains from this estimator are significantly more efficient than that of the Close-Close Estimator. The practical importance of this improved efficiency is that seven times fewer observations are necessary in order to obtain the same statistical precision as the Close-Close estimator[17]. The random variable volatility which is estimated has a tighter sampling distribution.

At this point one should also mention that due to the fact that extreme value estimators of realized volatility are derived using strict assumptions, they may likely tend to be biased estimates of realized volatility although being more efficient than the classical Close-Close estimator[18].

2.2 Implied Volatility

Implied volatility is the theoretical value which represents the future volatility of the underlying financial asset for an option as determined by today's price of the option. Implied volatility can be implicitly derived by inversion using option pricing models. When the market price of the option is known one can simply calculate the (local) volatility that would have been used in the option pricing model to give the observed option price taken into consideration. The most famous pricing model is the Black and Scholes Option Pricing Model[19]. First its derivation will be shown and then the calculation of it's implied (local) volatility function by Dupire[20]. Bruno Dupire showed that if the stock price follows a risk neutral random walk and if no-arbitrage market prices for European vanilla options are available for all strikes K and expiries T, then the implied (local) volatility used as a variable within the option price model, can be expressed as a function of K and T.

Due to the fact that empirical observations of options have shown that volatility does not remain constant as exercise price (strike) and expiries changes as assumed by Black and Scholes. Modern day Finance researchers have moved on to the next level of precision and have incorporated stochastic volatility into their models. The second part of this section will deal with such stochastic volatility models, in particularly the Heston Stochastic Volatility Model.

---

[17] See Garman M.B., M.J. Klass, 1980

[18] Li, K., D. Weinbaum, 2000, "The Empirical Performance of Alternative Extreme Value Volatility Estimators", Working Paper, Stern School of Business, New York.

[19] Black, F. and M. Scholes, 1973, "The pricing of options and corporate liabilities", Journal of Political Economy 81, 637-659.

[20] Dupire, B. 1994. "Pricing with a Smile". Risk Magazine, 7 18-20.

2.2.1 Black-Scholes and Local Volatility Model

Before the Black and Scholes partial differential equation (PDE) and its solution for an European call can be derived, certain assumptions have to be implemented.

1. There are no market restrictions
2. There is no counterparty risk and transaction costs
3. Markets are competitive
4. There are no arbitrage opportunities i.e. two identical assets cannot sell at difference prices; therefore there are no opportunities by market participants to make an instantaneous risk-free profit.
5. Trading takes places continuously over time
6. Stock price follow a Brownian motion i.e. stock prices are random.
7. Stock price follows a lognormal probability distribution
8. Interest rates are constant
9. In order to avoid complexity, dividend payments are not incorporated into the following analyses

Ito's Lemma can be used to manipulate random variables. It relates the small change in a function of a random variable to the small change in the random variable itself. In order to proceed with the derivation of the Black and Scholes formula on need to define the stochastic differential equation (SDE) of the form:

$$dX = A(X,t)dt + B(X,t)dW$$

where $A(X,t)$ is known as the drift term, $B(X,t)$ the volatility function and dW represents a Brownian motion. Thus if $f(X)$ be a smooth function, Ito's lemma says that:

$$df = B\frac{\partial f}{\partial X}dW + \left(A\frac{\partial f}{\partial X} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial X^2}\right)dt$$

Thus adding the variable t to $f(X)$ gives $f(X,t)$ and Ito's lemma says that:

$$df = B\frac{\partial f}{\partial X}dW + \left(A\frac{\partial f}{\partial X} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial X^2} + \frac{\partial f}{\partial t} \right)dt$$

Equation 7a: One Dimensional Ito's Lemma

Now if X, Y are SDEs:

$$dX = A(X,t)dt + B(X,t)d\hat{W}_1$$

$$dY = C(Y,t)dt + D(Y,t)d\hat{W}_2$$

whereby the two Brownian Motion instants have a correlation r , then for $f(X,Y,t)$ Ito's lemma says:

$$df = B\frac{\partial f}{\partial X}dW_2 + D\frac{\partial f}{\partial Y}dW_2 + \left(A\frac{\partial f}{\partial X} + C\frac{\partial f}{\partial Y} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial X^2} + rBD\frac{\partial^2 f}{\partial X \partial Y} + \frac{1}{2}D^2\frac{\partial^2 f}{\partial Y^2} \right)dt$$

Equation 7b: Two Dimensional Ito's Lemma

Considering the following SDE where the average rate of growth of the stock, also known as the drift = m , and volatility = s and both are constants. Let stock price = S then:

$$dS = Smdt + Ss\,dW$$

Equation 8: Stochastic Equation of Small Change in S

Suppose that $f(S)$ is a smooth function of S . So if S were to be varied by a small amount dS , then f would also vary by a small amount. Using the Taylor series expansion, one derives,

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + ...O(dS^3)$$

further generalizing this result and introducing the variable time to the function, we get $f(S,t)$. Imposing a small change on $f(S,t)$ one derives $f(S+dS, t+dt)$ which can be expanded using the Taylor Series Expansion to give:

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + ...O(dS^3)$$

Equation 9: Taylor Series Expansion of $f(S+dS, t+dt)$

In equation (8) dS represents a small randomly change in the variable S (stock price). Squaring that we get:

$$dS^2 = S^2 m^2 dt^2 + 2S^2 ms\, dtdW + S^2 s^2 dW^2$$

Equation 10: Squared Stochastic Equation of Small Change in S

If :

$$dW^2 \to dt, \text{ as } dt \to 0$$

then the third term in dS^2 is the largest for small dt and therefore dominates the other terms.

Therefore:

$$dS^2 = S^2 s^2 dt$$

Substituting the above result into equation (9) results in:

$$df = \frac{\partial f}{\partial S} dW + (Smdt + Ss\, dW) + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{2\partial S^2} S^2 s^2 dt$$

$$= Ss \frac{\partial f}{\partial S} dW + \left(Sm \frac{\partial f}{\partial S} + \frac{1}{2} S^2 s^2 \frac{\partial^2 f}{\partial S^2} + \frac{df}{\partial t} \right) dt$$

At this step the hedging portfolio is introduced into the model. In its simplest form hedging against price movemnets entails taking a long (short) position in an option contract while simultaneously taking a short (long) position in the underlying financial asset. This can reduce the risk of the portfolio. One important hedging strategy is delta hedging. The delta Δ of the option is defined as the change of the option price with respect to the change in the price of the underlyi ng financial asset.

Now Black-Scholes equation can be derived for an European option V with an arbitrary payoff $V(S,T) = \Psi(S)$. Forming a portfolio Π which is delta-hedged (according to definition given above $\Delta = \frac{\partial V}{\partial S}$) with the delta-factor $(let\ \frac{\partial V}{\partial S} = f)$ gives:

$$\Pi = V - fS$$

the delta-factor is constant and makes the portfolio risk-free. A change in the value of the portfolio can be represented as:

$$d\Pi = dV - fdS$$

Suppose a change in the stock price S satisfies the following stochastic differential equation (SDE):

$$dS = Srdt + Ss dW$$

where the drift term m is represented by the risk-free bank rate r and the volatility of the stock is equal to s .

Applying the Ito's lemma to V , one derives:

$$dV = Ss \frac{\partial V}{\partial S} dW + \left(Sr \frac{\partial V}{\partial S} + \frac{1}{2} S^2 s^2 \frac{\partial^2 V}{\partial S^2} + \frac{dV}{\partial t} \right) dt \text{ ,}$$

Therefore substituting values dV and dS into:

$$d\Pi = dV - f dS$$

one gets:

$$d\Pi = Ss \left[\frac{\partial V}{\partial S} - f \right] dW + \left(Sr \left[\frac{\partial V}{\partial S} - f \right] + \frac{1}{2} S^2 s^2 \frac{\partial^2 V}{\partial S^2} + \frac{dV}{\partial t} \right) dt$$

Equation 11: Small Change in Hedged Portfolio under Ito's Lemma

By substituting $\frac{\partial V}{\partial S} = f$ in equation (11) one derives a risk-free portfolio (risk-neutralization) without the Brownian motion term dW which makes the equation deterministic (no randomness):

$$d\Pi = \left(\frac{1}{2} S^2 s^2 \frac{\partial^2 V}{\partial S^2} + \frac{dV}{\partial t} \right) dt$$

Since this portfolio contains no risk it must earn the same as other short-term risk-free financial assets. Following the principle of no-arbitrage, portfolio Π must earn the risk-free bank rate r :

$$d\Pi = r\Pi dt$$

Equation 12: Risk-free Portfolio

substituting $\Pi = V - fS$ into equation (12) one gets:

$$d\Pi = r(V - \frac{\partial V}{\partial S} S)dt,$$

and combining $d\Pi = \left(\frac{1}{2} S^2 s^2 \frac{\partial^2 V}{\partial S^2} + \frac{dV}{\partial t} \right) dt$ with $d\Pi = r(V - fS)dt,$ and dividing by dt , then rearranging one derives the Black-Scholes linear parabolic partial differential equation:

$$\frac{dV}{\partial t}+\frac{1}{2}s^{2}S^{2}\frac{\partial^{2}V}{\partial S^{2}}+rS\frac{\partial V}{\partial S}-rV=0$$

Equation 13: Black-Scholes Partial Differential Equation

Considering a European vanilla option that has boundary conditions (payoffs):

$$V(S,T)=\begin{cases}\max(S-K,0)\rightarrow Calls\\ \max(K-S,0)\rightarrow Puts\end{cases}$$

The Black-Scholes PDE needs these boundary conditions in order to attain a unique solution. Deriving the explicit function of a European call $C(S,T)$ gives:

$$C=SN(d_1)-Ke^{-r(T-t)}N(d_2),$$

where,

$$N(d)=\frac{1}{\sqrt{2p}}\int_{-\infty}^{d}e^{-x^2/2}dx,$$

$$d_1=\frac{\ln(S/K)+(r+s^{2}/2)(T-t)}{s\sqrt{T-t}},\ \ d_2=d_1-s\sqrt{T-t},$$

Equation 14: Explicit Solution of Black-Scholes PDE for a European Call

where $N(d)$ is the standard normal cumulative distribution function. The fact that only when deriving an explicit solution of the Black-Scholes PDE a derivative product is specified through the use of the boundary conditions, reiterates the advantage of the Black-Scholes PDE in solving the pricing dilemma of several types of options.

The explicit solution of $C=SN(d_1)-Ke^{-r(T-t)}N(d_2)$, which give the value of the option can be used, along with a numerical method like the Newton-Raphson Method to estimate the unique implied volatility of an option with option value C. If one were to calculate different values of $C(S,T)$, i.e. always varying the strike K and expiration T one would observe a flat (constant) volatility surface along strikes and expirations as shown in figure 1 below.

But if real market data were to be used the volatility surfaces represented by the data would resemble that of figure 2. Financial markets exhibit several different patterns of volatility surfaces with varied strikes (skewness) and maturities (term structure). These patterns are known as the volatility smile or skew. Therefore the (Black-Scholes) implied volatility for an option can be considered as the constant volatility which when substituted in the Black-Scholes model *ceteris paribus* gives the observed market price of the option.

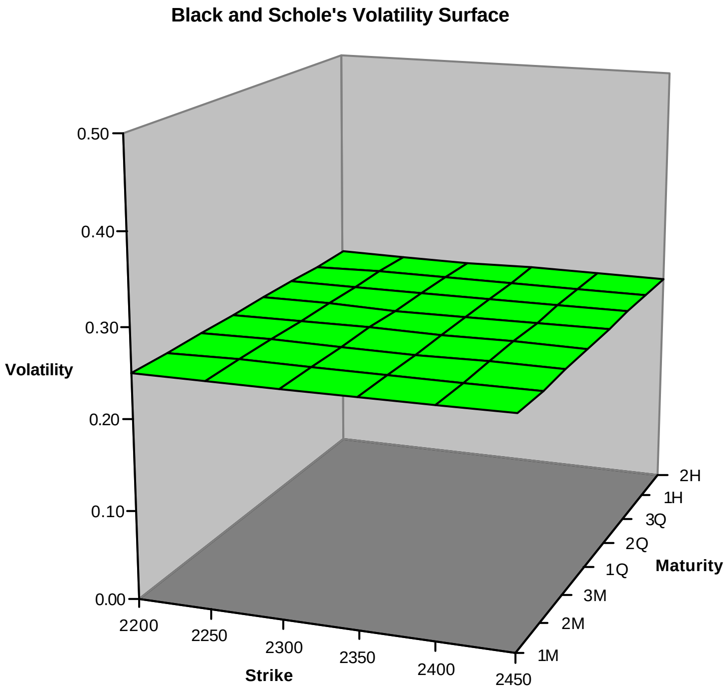


Figure1: Black and Schole's Volatility Surface

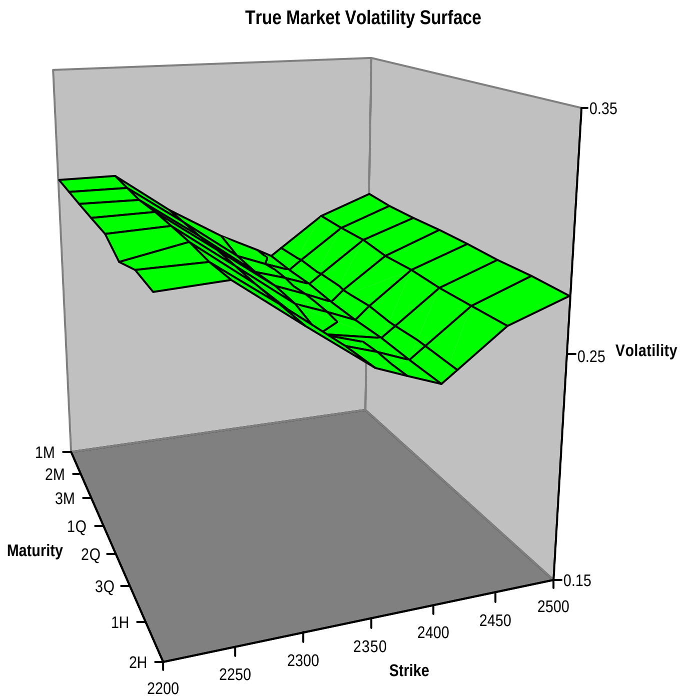


Figure 2: True Market Volatility Surface

Bruno Dupire[21] in his 1994 research paper proved that under the conditions of risk-neutral Brownian motion and no-arbitrage market prices for European vanilla options a local (implied) volatility $s_L(K,T)$ can be extracted by applying the Black-Scholes PDE to observed market prices.

Assuming that stock prices follow a risk-neutral random walk of the form:

$$dS = mdt + s(S,t)SdW,$$

where by $s \to s(S,t)$ becomes a local volatility (i.e. volatility is dependent on the strike and time), then an explicit solution of the Black-Scholes PDE for a vanilla European call option becomes dependent on the unknown local volatility function:

$$C = (S,t;K,T;s(S,t),r)$$

or expressed as a PDE:

$$\frac{\partial C}{\partial t} = \frac{1}{2}S^2 s^2(S,t)\frac{\partial^2 C}{\partial S^2} - r\left(S\frac{\partial C}{\partial S} - C\right)$$

If one was to inverse the European call function in order to solve for s , the implied volatility calculated would be a function of current stock price S and time t. But what is actually required is a local implied volatility as a function of the strike and expiration $s_L(K,T)$. So translating the call option into (K,T) -space results in a call function expressed as

$$C(S,t;K,T;s(K,T),r)$$

or expressed as a PDE:

$$\frac{\partial C}{\partial T} = \frac{1}{2}K^2 s^2(K,T)\frac{\partial^2 C}{\partial K^2} - r\left(K\frac{\partial C}{\partial K} - C\right)$$

Equation 15: Dupire's PDE Equation

Rearranging this equation results in a local (implied) volatility expression:

$$s^2(K,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} + C}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}}$$

Equation 16: Local Implied Volatility

[21] Dupire, B., 1994, "Pricing with a smile". Risk Magazine, 7, 18-20

Due to the fact that option prices of different strikes and maturities are not always available or insufficient, the right local volatility cannot always be calculated[22].

2.2.2 Stochastic Volatility

As shown in figure 2 the market volatility surface is actually skewed. The goal of a stochastic volatility model is to incorporate this empirical observation. This is implemented into the model by assuming that volatility follows a random (i.e. stochastic) process. The model which will be illustrated is the Heston Model[23]. This model is very popular because of two factors. Firstly the Heston Model allows for the correlation between asset returns and volatility and secondly it has a semi-analytical pricing formula.
In deriving the stochastic volatility model one assumes the usual geometric Brownian Motion SDE where volatility s is represented as the square root of the variance v.

This gives a stochastic differential equation of the form:

$$dS = Srdt + S\sqrt{v}d\hat{W}_1$$

where the variance v is now stochastic and follows its own random process:

$$dv = ((w - Vv) - \Lambda)dt + xv^g d\hat{W}_2$$

whereby, x is the volatility of volatility and r is the correlation between the two Brownian processes $d\hat{W}_1$ and $d\hat{W}_2$. This relationship implements the mean-reversion characteristic of volatility into the model. The real world drift is represented by $(w - Vv)$ and Λ symbolizes the market price of volatility. This relates how much of the expected return of the option under consideration is explained by the risk (standard deviation) of v. Let $\Lambda = lv$, which makes it proportional to variance and the real world drift were to be re-parameterized in the form:

$$(w - Vv) = k(q - v)$$

one gets a transformed SDE:

$$dv = (k(q - v) - lv)dt + xv^g d\hat{W}_2$$

Equation 17: Transformed Volatility SDE

[22] Derman, E., and Kani,I., Riding on a smile, Risk, 7 (1994), pp. 32--39

[23] Heston, S.L., 1993, “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.”, The Review of Financial Studies, Volume 6, Issue 2, 327-343.

Where k is the mean-reverting speed and q the long term mean. All parameters are constants. Forming a portfolio in which volatility risk must be hedged can be done by holding a position in a second option. So the portfolio would consist of a volatility dependent option V, a long or short position in a second option U as well as the underlying S. Therefore the hedged portfolio can be represented as:

$$\Pi = V - f_1 S - f_2 U$$

The small change in the value is illustrated as:

$$d\Pi = dV - f_1 dS - f_2 dU$$

and a small change in the portfolio in dt time is:

$$d\Pi = adS + bdv + cdt$$

where:

$$a = \frac{\partial V}{\partial S} - f_1 - f_2 \frac{\partial U}{\partial S}$$

$$b = \frac{\partial V}{\partial v} - f_2 \frac{\partial U}{\partial v}$$

$$c = \left(\frac{\partial V}{\partial t} + \frac{1}{2} S^2 v \frac{\partial^2 V}{\partial S^2} + rSxv^{2g+\frac{1}{2}} \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} x^2 v^{2g} \frac{\partial^2 V}{\partial v^2} \right) - f_2 \left(\frac{\partial U}{\partial t} + \frac{1}{2} S^2 v \frac{\partial^2 U}{\partial S^2} + rSxv^{g+\frac{1}{2}} \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} x^2 v^{2g} \frac{\partial^2 U}{\partial v^2} \right)$$

Equation 18: Change in time dt of Portfolio with stochastic volatility

In order to neutralize the risk in the portfolio, the stochastic components ($a = b = 0$) of risk are set to zero. Therefore rearranging the hedge parameters will give:

$$f_1 = \frac{\partial V}{\partial S} - f_2 \frac{\partial U}{\partial S}$$

which will eliminate the dS term in equation 18 and

$$f_2 = \frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}}$$

to eliminate the dv term in equation 18. The non-arbitrage condition of this portfolio is represented by:

$$d\Pi = r\Pi dt$$

and substituting Π in the equation gives:

$$d\Pi = r(V - f_1 S - f_2 U)dt$$

which simply signifies that the return on a risk-free portfolio must be equal to the risk-free bank rate r in order to prevent arbitrage possibilities.

Introducing equation 18 into the risk-free, non-arbitrage portfolio and collecting the V term on one side and all U on the other side will give an arbitrary pair of derivative contracts. This can only occur when the two contracts are equal to some function depending only on S, v, t.

Therefore let both derivative contracts be represented by $f(S,v,t)$, whereby f is the real world drift term less the market price of risk (see equation 17):

$$f(S,v,t) = (k(q-v) - lv)$$

then the PDE from the Heston Model is:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}S^2 v \frac{\partial^2 V}{\partial S^2} + rSxv\frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}x^2 v \frac{\partial^2 V}{\partial v^2} + Sr\frac{\partial V}{\partial S} + (k(q-v) - lv)\frac{\partial V}{\partial v}\right) = rV$$

This can also be derived from the two dimensional Ito's lemma equation (equation 7b).

The Heston's model is superior in the theory in comparison to the Black-Schole's Model because its assumption of a variable volatility mirrors that of market and empirical observations. But like the Black-Schole's it falters in some cases due to the general assumptions within the model. For instance due to the fact that within the Heston's Model assets prices are assumed to be continuous, large price changes in either direction (i.e. jumps) are not allowed in the process assumed by the model. In reality price jumps are a natural phenomenon, for example during economic shocks.

Another important limitation is that of the interest rate which is assumed to be constant. In the real world interest rate do change over time and maturity. In the literature it is widely suggested that the volatility of the underlying is negatively correlated with interest rates. If this is true then the implementation of a stochastic interest rate and arbitrary correlation between interest rates and volatility into the Heston's Model could possibly improve its estimations dramatically.

2.3 Discrete Time Model: GARCH Model

The aforementioned models possess the assumption of continuous time. Although such models provide the natural framework for an analysis of option pricing, discrete time models are ideal for the statistical and descriptive analysis of the distribution of volatility. One such class of discrete time models is the autoregressive conditional heteroskedastic (ARCH) models which were introduced by Engle[24]. An ARCH process is a mean zero, serially uncorrelated process with non-constant variance conditional to the past, but with a constant unconditional variance. The ARCH models have been generalized by Bollerslew[25] in the generalized ARCH (GARCH) models. The GARCH (1,1) models seem to be adequate for modeling financial time series[26]. As result the GARCH (1,1) will be the only discrete time model which will be introduced in this section.

GARCH stands for Generalized Autoregressive Conditional Heteroskedasticity. Heteroskedasticity can be considered as the time varying characteristic of volatility (square root of variance). Conditional means a dependence on the observations of the immediate past and autoregressive describes a feedback mechanism that incorporates past observations into the present.

Therefore one can conclude that GARCH is a model that includes past volatilities (square root of variance) into the estimation of future volatilities. It is a model that enables us to model serial dependence of volatility. GARCH modeling builds on advances in estimating volatility. It takes into account excess kurtosis (fat-tailed

[24] Engle, R.F., 1982, "Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation", Econometrica.
[25] Bollerslew, T., 1986, Generalized Autoregressive Conditional Heteroskedasticity", Journal of Econometrics, Vol. 3, 307-327.
[26] Duan, J.C., 1990, "The GARCH Option pricing Model, unpublished manuscript, McGill University.

distribution) and volatility clustering, two important characteristic of real market volatility observations.

Financial data has shown that the variance seems to be varying from time to time and usually a large movement in both directions seems to be followed by another. This is termed volatility clustering. Unlike the assumed normal distribution of log returns in asset prices, empirical data of such returns have depicted fat-tailed distributions. Tail thickness can be measured in kurtosis (the fourth moment) with the kurtosis of normal distribution being at a value of 3.

However market data have possessed thicker tails, i.e. a kurtosis greater that 3. The GARCH[27] models have been constructed to capture these features.

Let a series of assets returns r_t which are conditionally modeled be represented as:

$$r_t | I_{t-1} = m_t + e_t$$

I_{t-1} denotes the information available in $t-1$ time and the conditional mean m_t contains a constant, some dummy variables to capture calendar and possibly autoregressive or moving average term. The stochastic change e_t is expressed for a GARCH class of models in terms of a normal distributed variable as:

$$e_t | I_{t-1} \sim N(0, s_t^2)$$

where s_t^2 is the time-varying variance. Different constellations of s_t^2 as a deterministic function of past observations and past conditional variances give rise to several kinds of GARCH-type models. Considering the conditional variance s_t^2 as a linear function both of p past squared innovations and q lagged conditional variances, one derives the standard GARCH (p,q) model introduced by Bollerslev (1986).

$$s_t^2 = w + \sum_{i=1}^{p} a_i e_{t-1}^2 + \sum_{j=1}^{q} b_j s_{t-j}^2 \equiv w + a(L) e_t^2 + b(L) s_t^2$$

Equation 19: Conditional Variance of a standard GARCH (p,q)

where L denotes the lag operator. Imposing the restriction $b_j = 0$ for any j, gives the original ARCH (p) model from Engle (1982). The ARCH (1) model is a special case of the GARCH $(1,1)$ with $b_j = 0$. Autoregressive Conditional Heteroskedasticity was first

[27] Bollerslev, T., 1986: “Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics”, 31, 307–327.

introduced by Robert Engle[28] in 1982 who later went on to become a Nobel Prize Laureate in 2003.

2.4 Forecasting Abilities of Volatility Estimators

It has been mentioned above that financial market volatility has been known to show fat tails distribution, volatility clustering, asymmetry and mean reversion. Some researches have shown that volatility measures of daily and intra-day returns possess long data memory[29]. These results are relevant because they infer that a shock in the volatility process of the likes of jumps have long lasting implications on estimations.

The mean reversion of volatility creates some problems by the selection of the forecast horizon. In their paper Andersen, Bollerslev and Lange[30] (1999) empirically showed that volatility forecast accuracy actual improves as data sampling frequency increases relative to forecast horizon. Furthermore Figlewski[31] (1997) found out that forecast error doubled when daily data, instead of monthly is used to forecast volatility over two years. In some cases where very long horizon are used, e.g. over 15 years, it was better to calculate the volatility estimates using weekly or monthly data, due to the fact that volatility mean reversion is difficult to adjust using high frequency data. In general, model based forecasts lose on quality when the forecast horizon increases with respect to the data frequency.

In their paper Poon and Granger[32] (2002) reviewed the results of 93 studies on the topic of volatility forecasting. They came to the conclusion that implied volatility estimators performed better than historical and GARCH estimators, with historical and GARCH estimators performing roughly the same. They went on further to say that the success of the implied volatility estimators does not come as a surprise as these forecasts use a larger and more relevant information set than the alternative methods as they use option prices, but also reiterated that implied volatility

[28] Engle, R. 1982, "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, Econometrica 52, 289-311.
[29] Granger, C.W.R., Z. Ding and S. Spear, 2000, "Stylized facts on the temporal and distributional properties of absolute returns, Working paper, University of California, San Diego
[30] Andersen, T., T. Bollerslev and S. Lange, 1999, "Forecasting financial market volatility: Sample frequency vis-à-vis forecast horizon", Journal of Empirical Finance, 6, 5, 457-477.
[31] Figlewski, S., 1997, "Forecasting volatility", Financial Markets, Institutions and Instruments, New York University Salomon Center, 6, 1, 1-88.
[32] Poon, S.H. and C. Granger, 2002, "Forecasting Volatility in Financial Markets: A Review"

estimators are less practical, not being available for all asset classes. They concluded that financial volatility can clearly be forecasted. The main issues are how far into the future can one accurately forecast volatility and to what extent, can volatility changes be predicted. The option implied volatility, being a market based volatility forecast has been shown to contain most information about future volatility. Historical volatility estimators performed differently among different asset classes but in general, they performed equally well as GARCH models.

3. Volatility Trading and the New Volatility Indices of the Deutsche Boerse

3.1 Volatility Trading

Over the recent decades volatility has gained in popularity as a tradable instrument in financial markets, especially in the "over the counter" markets (OTC). This growth in interest is mainly due to several of its basic characteristics. Firstly volatility tends to grow in periods of uncertainty and therefore acts as a gauge for uncertainty which reflects the general sentiments of the market. Secondly its negative correlation to its underlying and its statistical property of mean reversion equip volatility with characteristics which are quite valuable to financial market participants. Lastly implied volatility tends to be higher than realized volatility thus creating opportunities for speculative trading. This reflects the general aversion of investors to be short on option volatility. Therefore a risk premium is paid to the investor to remunerate him for going short on implied volatility.

Volatility is one of the most important financial risk measures that need to be monitored (because of its use as an information tool for researchers, warrants issuer and users) and hedged, due to the main fact, that all market participants are somehow influenced directly or indirectly by volatility levels and its movements.

Over the years various strategies have been developed by financial practitioners to capture volatility. One such strategy is the use of straddles. This is the most common option strategy designed to capture the volatility of an underlying. In recent times an OTC market for trading with volatility and variance swaps has picked up. This has made it possible to trade in pure volatility.

One can recognize three generic types of volatility traders on the market, namely the directional traders, spread traders and volatility hedgers. Directional traders speculate on the future levels of volatility, while spread traders guess on either the spread between implied and realized volatilities or the spread between the volatility levels of say two indices. On the other hand volatility hedgers like hedge funds managers will want to cover their short volatility positions.
There are several ways to be short on volatility. A passive index tracker is implicitly short volatility since his rebalancing costs increase with increasing volatility. Benchmarked portfolio managers have an increasing tracking error with increasing market volatility which makes their portfolio implicitly short volatility. Lastly equity fund managers are implicitly short volatility due to the existence of a negatively correlated relationship between volatility and underlying returns.

3.1.1 Straddles

A straddle strategy entails the purchasing of both the call and put options on the same strike. This means that the purchaser is not speculating on a directional movement but simply on a movement regardless in whatever direction, hence this strategy relies on the volatility of the underlying to make money.

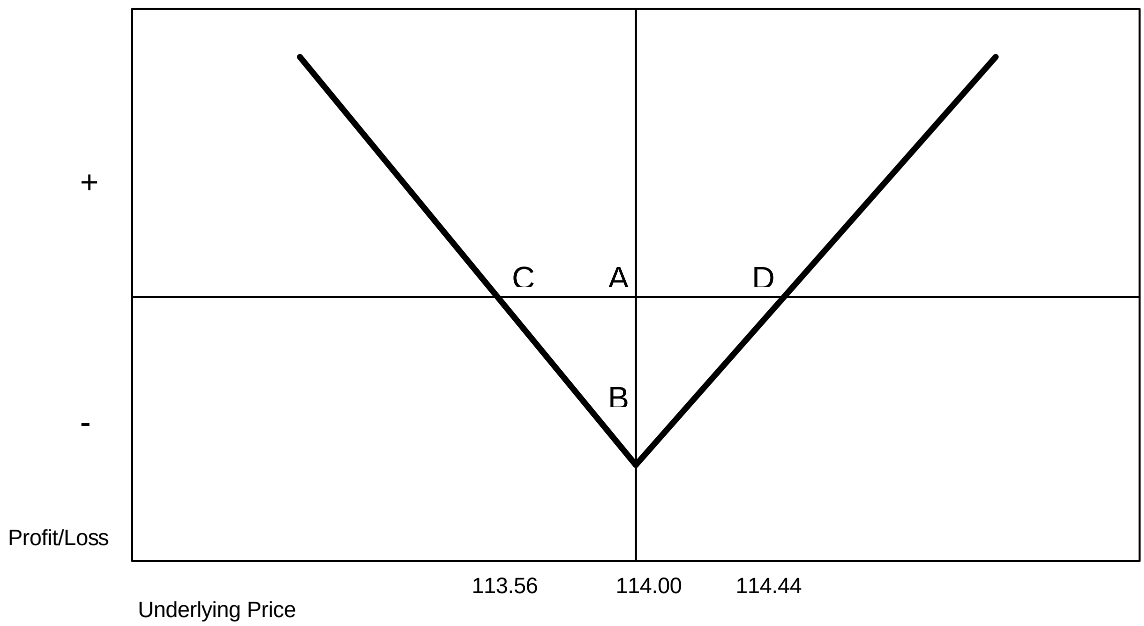


Figure 3: Straddle

Lets assume that the September future on the Bund (i.e. the future on German 10-year bonds) is trading at 114.00 on the last trading day. The 114.00 straddle is trading at 44 ticks. If the market remains at 114.00 for the whole day, the option owner has paid 44 ticks for a straddle that is worth 0 on expiry; as the market expires at 114.00, neither the calls nor the puts are in-the-money an as such, the option holder will lose all his premium. Looking at figure 3 above, the distance between A and B is the premium paid to purchase the straddle. At this point, the straddle is exactly at-the-money. If the market moves up, the 114.00 calls will be in-the-money and the option holder will start to earn back some of the 44 ticks he paid for the straddle. If the market moves down, the puts will be in-the-money and, once again, some of the 44 ticks paid out will be earned back. Therefore the option holder is speculating purely on volatility. At point C (113.56) and point D (114.44), the puts or calls respectively have made enough to cover the cost of the straddle. These breakeven points are 44 ticks away from the strike (114.00).

3.1.2 Swap Trading: Volatility and Variance

Through the use of volatility and variance swaps, traders are synthetically exposed to pure volatility. In reality volatility and variance swaps resemble more closely a forward contract than a swap whose payoff are based on the realized volatility of the underlying equity index like the EuroStoxx 50. Unlike such option-based strategies like that of the straddle or hedged puts or calls, these swaps have no exposure to the price movements of the underlying asset. A major negative aspect of using option-based strategies is that once the underlying moves, a delta-neutral trade becomes inefficient. Re-hedging becomes inevitable in order to maintain a delta-neutral position by market fluctuations. The resulting transaction and operation costs of re-hedging general prohibit a continual hedging process. Therefore a residual exposure of the underlying asset ultimately occurs from option-based volatility strategies. Although volatility and variance swaps serve the same purpose, they are not exactly identical. There are some important aspects of both which make them unique. One such aspect is that of their payoff functions. While volatility swaps exhibit a linear payoff function with respect to volatility, variance swaps on the hand have non-linear (curvilinear) payoff functions. Furthermore volatility swaps are much more difficult to price and risk-managed.

As mentioned above volatility swaps are forward contracts on realized historical volatility of the underlying equity index (e.g. EuroStoxx 50). The buyer of such a contract receives a payout from the counterparty selling the swap in case the volatility of the underlying realized over the swap contract's life exceeds the implied volatility swap rate quoted at the interception of the contract. The payoff at expiration is based on a notional amount times the difference between the realized volatility and implied volatility:

$$Payoff = €_{notional} \times (s_{realized} - s_{implied})$$

All volatilities are annualized and quoted in percentage points. The notional amount is typically quoted in Euros per volatility percentage point. Take for instance a volatility swap with a notional amount of € 100,000 per volatility percentage point and a delivery price of 20 percent. If at maturity the annualized realized volatility over the lifetime of the contract settled at 21.5 percent then the owner would received:

$$Payoff = €100{,}000 \times (21.5 - 20) = €150{,}000$$

The implied volatility is the fixed swap rate and is established by the writer of the swap at the time of contractual agreement.
The general structure and mechanics of a variance swap are similar to that of a volatility swap. The main dissimilarity between the two volatility derivatives is that realized and implied variances (volatility-squared) are used to calculate the pay-off and not realized and implied volatilities.

$$Payoff = €_{notional} \times (s^2_{realized} - s^2_{implied})$$

As mentioned above, the use of variance instead of volatility results in a nonlinear payoff. This means loss and gains are asymmetric. Therefore, there is a larger payoff to the swap owner when realized variance exceeds implied variance, compared to the losses incurred when implied variance exceeds realized variance by the same volatility point magnitude. The swap rate is essentially the variance implied by a replicating portfolio of puts and calls on the index. The synthetic portfolio is so constructed that its value is irresponsive of stock price moves. This combination of calls and puts is a weighted combination across all strikes (i.e. from zero to infinity), with the weights consisting of the inverse of the square of the strike level. Prices of less liquid or non-traded options are estimated via interpolation and extrapolation. All the options within the portfolio possess the same expiration date as the variance swap contract. Therefore, the variance implied from the market value of this portfolio becomes the swap rate of the volatility derivative. At expiration, if the index's realized variance is below the swap rate, then the swap holder makes a payment to the swap writer. The opposite payment flow occurs, if the swap rate is higher than the realized variance at expiration.

There are some trading strategies which can be applied to volatility or variance swaps. It has been empirically shown that implied volatility is often higher that the volatility realized over the lifetime of the option[33]. Given the structure of these derivatives, going short on variance swaps can be used to capture the difference between historical and implied volatility. Therefore a trader can sell a variance swap and earn profits as the contract expires. Another strategy is to use variance swaps to execute stock index spread trading. Such a strategy can be implemented using a short variance swap on an equity index (EuroStoxx 50) which is then partially hedged by a long swap on another index (S&P 500). This spread has a payoff based on the difference between the realized volatility (or variance) of the two indices.

At inception, the swap contract will have a zero market value, but throughout the life of the contract the market value of the swap is primarily influenced by changes in the volatility surface for options of similar maturities based on the remaining life of the variance swap.

3.2 The Methodologies of the Volatility Indices

Implied volatility at the Deutsche Börsewill be calculated in future using two different types of methodologies. An old concept, which will continue to be used to calculate the volatility of the DAX® (old VDAX) and the new model which will be introduced to calculate the volatility of the new VDAX, the VSTOXX (volatility of the EuroStoxx 50®) and the VSMI (volatility of the SMI®).

[33] Fleming J., 1998, "The Quality of Markets Forecasts Implied by S&P100 Option Prices", Journal of Empirical Finance, 5, 317-345

3.2.1 The Old Methodology

Computing volatility using the old model requires three components. Firstly, an option model, secondly the values of the model's parameters, except that of volatility and lastly, an observed price of the option on the index. The option model used here in the calculation is based on the Black-Scholes Option Pricing Model[34] applied to a European call option. There is a slight modification to the original model which relates to the underlying's valuation. The Forward index level is used instead of the underlying's present index level. This can be expressed as:

$$F = Se^{rt}$$

Substituting the forward index level (F) for the index level (S) in the equation (14) results in the following expressions below:

$$C = e^{-r(T-t)}(FN(d_1) - KN(d_2)), \qquad [1]$$

$$P = e^{-r(T-t)}(KN(-d_2) - FN(-d_1)), \qquad [2]$$

where,

$$N(d) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{d} e^{-x^2/2} dx, \qquad [3]$$

$$d_1 = \frac{\ln(F/K)}{s\sqrt{T-t}} + \frac{s\sqrt{T-t}}{2}, \qquad [4]$$

$$d_2 = d_1 - s\sqrt{T-t}, \qquad [5]$$

Equation 20: Explicit Solution of Black-Scholes using the Forward Index Level

whereby:

C, Call price

P, Put price

F, Forward price of the index level

$T-t$, Time to expiration

r, Risk-free interest rate

s, volatility of the option

$N(...)$, Normal distribution function

[34] See section 2.2.1.

The refinancing factor R is expressed as:

$$R = e^{r(T-t)} \qquad [6]$$

When expressions [1] and [2] are re-parameterized to make them dimensionless, results in the following transformations:

$v = \dfrac{s\sqrt{T-t}}{2}$ [7], generalized volatility

$c = \dfrac{CR}{\sqrt{FK}}$, [8], generalized call price

$p = \dfrac{PR}{\sqrt{FK}}$, [9], generalized put price

$f = \dfrac{F}{\sqrt{FK}}$, [10], generalized forward price

$u = \ln(f)$, [11], logarithmic of generalized forward index level

Therefore the resulting generalized call and put prices can be represented as:

$$c = e^{+u} N(\frac{u}{v} + v) - e^{-u} N(\frac{u}{v} - v), \ [12]$$

$$p = e^{-u} N(-\frac{u}{v} + v) - e^{+u} N(-\frac{u}{v} - v), \ [13]$$

Equation 21: Generalized Call and Put Formulae

These transformations create expressions of the call and put prices, which are expressed as functions of the forward index level (u) and volatility (v). These option price representations are the basis for the calculation of the volatility using the old methodology. The old methodology measures implied volatility using the at-the-money (ATM) option of the index. The implied volatility is numerically extracted from the ATM option price using the transformed Black-Scholes Option Pricing Model expressed above [35]. A draw back to this methodology is that it's computationally intensive.

The calculation of volatility using the old Methodology occurs in one minute intervals, whereby the respective best bid and best ask of all index options and future contracts listed on Deutsche Börseare extracted from the stream of data generated by the Eurex system. The option prices extracted are subject to a filtering process in which all one sided market option (i.e. either possessing only a bid or ask) are filtered out. Option with neither a bid nor ask are also automatically filtered out. Another filter verifies whether the bid/ask spread of each remaining option satisfy the criteria of

[35] see expressions [12] and [13].

staying within the maximum quotation spreads established for Eurex market-makers. Accordingly the maximum spread must not exceed 15% of the bid quote, with in the range of 2 basis points to 20 basis points[36].

The next step in this process is to calculate the mid-price for the filtered options and futures prices. Therefore for each maturity i and exercise j, the mid-prices of the bid b and ask a are calculated as follows:

$$C_{ij} = \frac{C_{ij}^{a} + C_{ij}^{b}}{2}, \text{ [14]}$$

$$P_{ij} = \frac{P_{ij}^{a} + P_{ij}^{b}}{2}, \text{ [15]}$$

$$F_{i} = \frac{F_{i}^{a} + F_{i}^{b}}{2}, \text{ [16]}$$

The corresponding interest rate which matches the time to expiration of the index option is derived through the use of linear interpolation. The two nearest interest rates $r(T_K)$ and $r(T_{K+1})$ (e.g. 1 week and 1 month Euribor rates) to the time to expiration T_i of the option under consideration and their respective time to expirations T_K and T_{k+1}, are interpolated to derive an approximation of the interest rate to be used in the calculation of the index. This is shown below:

$$r_i \equiv r(T_i) = \frac{T_{k+1} - T_i}{T_{k+1} - T_k} r(T_k) + \frac{T_i - T_k}{T_{k+1} - T_k} r(T_{k+1}), \text{ [17]}$$

Equation 22: Interpolation of interest rates

where,

$$T_k \leq T_i < T_{k+1}, \text{ [18]}$$

The information gained is then used in the calculation of the refinancing factor R_i using the relation in expression [6].

The determination of the forward price can occur in two distinctive steps. The first step entails calculating the preliminary forward prices of the index using the options' remaining time to expiration. If a future on the index under consideration has a matching time to expiration with the option on the index and is also quoted within the given maturity period of the option, then the mid-price of the future is used as the final forward price. The complexity of the determination of the forward price of the index price increases in cases where there are no index futures present in which the time to

[36] refers to old market making model which is no longer in use

expiration matches. In these cases, no forward price is then available in the Eurex system for the given index options expiry month. In such a case, a forward price is calculated in two steps. Firstly a preliminary forward price F' is estimated by way of linear interpolation, using those futures that have not been filtered out and are quoted around the time to expiration under consideration. If interpolation is not available due to the fact that no future with a longer remaining time to expiration is quoted and available, then extrapolation based on the longest available futures contract is used to calculate the preliminary forward price. The preliminary forward price calculated that way defines the preliminary at-the-money point. Only those option series j within a given expiry month, whose exercise prices are close to the preliminary forward price are taken into account in the next step of the calculation process.

For expiry months, where a preliminary forward prices was calculated by means of interpolation or extrapolation, the final forward price is now determined from the option prices, using the put-call parity method. For this purpose, pairs of calls and puts with the same exercise price are created.

Around the preliminary at-the-money point, a range of sixteen options is determined, i.e. the pairs of puts and calls of each of the four nearest exercise prices above and below this point. If no two pairs are simultaneously quoted within this range, the final forward price and therefore a current sub-index value cannot be determined. In such a case, if there is already an existing sub-index, this existing sub-index will continue to be used. If there are two or more pairs, every valid pair will be used in the calculation process. The reason for restricting to only eight exercise prices is to elude any series from the forward price calculation (using the call-put parity) which are either quoted not frequently enough or possess too wide a spread between bids and asks.

The calculation of the final forward price is expressed below:

$$F_i = \frac{1}{N} \sum_{C,P} \left[(C_{ij} - P_{ij}) R_i + K_j \right], \text{ [19]}$$

The expression above illustrates that the refinancing factor R_i and the forward price F_i have been established for every expiry month. The generalized, empirical option prices are calculated from the adjusted call and put prices according to the relations denoted in expression [8] and [9] above, using the exercise prices K_j.

As soon as the final forward price for a given time to expiration is determined, implied volatilities are calculated for all individual options which are relevant to this time to expiration and have not been filtered out. Since the generalized and adjusted option pricing formula, derived from the Black-Scholes Option Pricing Model cannot be directly solved for volatility, an iteration method is used to estimate the required value.

The starting value for the generalized volatility v [7] is set at 0.015. The theoretical generalized option price calculated using this value is compared to the market price of the option. Applying the Gauss-Newton method, a new generalized volatility v_i is gradually determined and used as the starting value for each successive iterative step. Upon attaining a given degree of accuracy (i.e. when v_i and v_{i+1} only differ from each other by 0.000003), the iteration process is stopped, yielding the option's generalized implied volatility. This is the value where the theoretical option price, calculated on the basis of that value, is almost identical to the market price of the option.

Before the calculation of a sub-index V_i can be carried out, the range of four (4 pairs calls and puts) option for the estimation process must be determined, this time, around the final forward price (calculated using the call-put parity), or the index futures price. Implied volatilities of each of the four options are weighted according to the distance of their exercise prices from the forward or futures price. Those four options selected have to be paired with the same exercise price, two higher and two lower than the calculated final forward price. Furthermore, how actual the options are, is given priority over the distance from the forward price. Accordingly, if there are current volatilities derived from the pair of options higher than the forward price, then the strike closer to the at-the-money point will be used. However, if the volatility of the more distant of the two higher options represents the more current information, then this strike (consisting of a call and a put) will be chosen.

For the interest rates determination, calculation is based on the values of the respective previous trading day until these interest rates are updated. With the index futures and options, this is different. In order to avoid volatility fluctuations, which are caused by changes in the index level from one day to the other, no previous day's values are used.

The dissemination of a sub-index requires the availability of certain data: firstly, it requires a forward price for the equity index with the same time to expiration as the sub-index; this value results directly from the index futures prices, or it is calculated. Around this forward price, defining the at-the-money point, the four individual volatilities used to calculate the index must be available. For this purpose, these volatilities do not need to have been traded simultaneously. As soon as the forward price as well as best bid and best ask prices, which have not been filtered out, are available for an option, its volatility can be determined. If the current data situation does not allow for a recalculation of this option's volatility in the subsequent calculation process, the last calculated value will continue to be used.

$$V_i = \frac{(K_h - F_i)*(v_l^{Put} + v_l^{Call}) + (F_i - K_l)*(v_h^{Put} + v_h^{Call})}{2(K_h - K_l)} \quad [20]$$

whereby:

V_i	=	Sub-index i, i.e. generalized volatility of time to expiration i
F_i	=	Forward or futures price of the corresponding time to expiration i
v	=	Implied volatility of an individual option
K	=	Strike price of the option

Subscripts h and l indicate whether reference is made to the higher or lower neighbouring exercise price.

If the final forward price's relevant strike price alters from one calculation to the next, and if the volatilities actually required for the calculation of the sub-index are not yet available for this new strike, the index is recalculated all the same – provided that both volatilities are available for at least one pair of the new neighbouring strike prices. In this case, only one volatility value is present for a given strike; this volatility is also used to estimate the missing fourth. If there are no such volatilities at all, the sub-index is determined as the average of the two existing volatilities. Again, how actual the data is, takes precedence over closeness to the forward price.

These sub-indices typically have no fixed remaining time to expiration, and will eventually expire. The objective behind the main volatility index is to construct a volatility index with a rolling, fixed time to expiration. This is attained through the interpolation of the nearest two sub-indices to the fixed remaining time to expiration. The related sub-indices V_i and V_{i+1} have been determined on the assumption that volatility is constant and the Black-Scholes Model is applicable. However, if volatilities s_i and s_{i+1} are not identical, and if, for example, s_{i+1} is greater than s_i, then market participants are obviously assuming higher average price fluctuations in the index for the far short term than for the near short term. Of course, in such a case, the statistical distribution of the equity index price fluctuations is no longer Gauss-shaped and, therefore resulting that the Black-Scholes Model is no longer appropriate to reflect this issue. However, the construction of the volatility index using this old methodology aims at adhering to the Black-Scholes Model all the same.

Based on the various assumptions stated above, volatility V over the fixed remaining time to expiration T is equal to the following:

$$V = \sqrt{\frac{T_{i+1} - T}{T_{i+1} - T_i} V_i^2 + \frac{T - T_i}{T_{i+1} - T_i} V_{i+1}^2}.$$

Equation 23: Linear Interpolation of variance

whereby: $T_i < T < T_{i+1}$

The equation above illustrates a linear interpolation of variances. By annualizing the interpolated volatility V one derives the main volatility index under the old methodology:

$$s = 2V \big/ \sqrt{T}$$

Equation 24: Volatility index using old methodology

3.2.1 The New Methodology

The new methodology used by the Deutsche Börse to calculate implied volatility utilizes the implied volatility derived from an at-the-money option and traded out-of-the money options of calls and puts per maturity series. This aims at making pure volatility tradable – i.e. the index should be able to be tracked by a portfolio which does not react to price fluctuations, but only to changes in volatility. But this is not directly achieved through volatility, but rather through variance or volatility squared. So, instead of only using implied volatilities around the at-the-money point, as with the "old" methodology, the new methodology also considers implied volatilities of out-of-the-money options of a given time to expiration.

In contrast to the old methodology, which is computationally intensive, the new model does not extract the implied volatility from an option pricing model like the Black-Scholes Option Pricing Model. The new methodology only involves summations over option prices and is thus computationally less demanding. The volatility index derived, measures the square root of the implied variance using index option prices of the equity index under consideration traded at Eurex.

Beside the continuous calculation of the main index with a rolling fixed maturity of 30 days, there are eight sub-indices measuring the volatility of the first eight option expiries having (1, 2, 3, 6, 9, 12, 18 and 24 months) to go at inception. The main index is determined by interpolation of two sub-indices which are nearest to a fixed remaining lifetime of 30 days. Therefore the calculation process of the new methodology can be divided into three main steps, mainly the data preparation and extraction, the calculation of the sub indices and finally the calculation of the main indices.

Before the data processing begins, the data are collected via snapshots every minute.

This includes:

- the best bid, ask, last prices as well as the settlement prices of all the equity index options.
- EONIA – the Euro-Overnight-Index-Average as overnight interest rate
- EURIBOR - the European Interbank Offered Rates as money-market interest rate for 1, 2, ... 12 months (calculated once a day, 11.00 a.m. CET, by the European Banking Federation)
- REX - the yield of the sub index with a maturity of 2 years (calculated on the basis of exchange-traded prices) as longer-term interest rate.

EONIA, EURIBOR and the yield of the REX are all risk-free interest rates.

Index Name	Period	Code	ISIN
EONIA	1 day	EU1D	EU0009659945
EURIBOR 1 month	1 month	EU1M	EU0009659937
EURIBOR 2 months	2 months	EU2M	EU0009652841
EURIBOR 3 months	3 months	EU3M	EU0009652783
EURIBOR 4 months	4 months	EU4M	EU0009652858
EURIBOR 5 months	5 months	EU5M	EU0009652866
EURIBOR 6 months	6 months	EU6M	EU0009652791
EURIBOR 7 months	7 months	EU7M	EU0009652874
EURIBOR 8 months	8 months	EU8M	EU0009652882
EURIBOR 9 months	9 months	EU9M	EU0009652890
EURIBOR 10 months	10 months	EU10	EU0009652908
EURIBOR 11 months	11 months	EU11	EU0009652916
EURIBOR 12 months	12 months	EU12	EU0009652809
REX 2-YEAR (PRICE INDEX)	2 years	REX2	DE0008469149

Figure 4: Table of interest rates

The option prices are subject to a data filtering process. For this purpose, options with a one-sided market, i.e. with a bid or an ask price only, and options with neither a bid nor an ask price, are disregarded. Another filter verifies the prices of the remaining quotes, checking if the bid/ask spread is within the maximum spreads for market makers at Eurex. The maximum spread restricts the quote to be within 15% of the bid quote, subject to a minimum of 2 points and a maximum of 20 points[37] for index expiry dates up to two years.

Example:

Bid = 45.32 and Ask = 54.3

Max. Spread: 45.32 * 0.15 = 6.798. Therefore, bid and ask illustrated above are discarded.

If the Eurex® activates the option „Fast Market“ in times of very hectic trading, Market Makers are allowed to double their maximum spreads. In that case the maximum spreads listed above are doubled for the equity index.

Yet a third filter checks if bid, ask, settlement and last prices have a minimum value of 0.5 Index points. This is the specified cut off point for the not yet filtered out options, which are too far out-of-the-money and are therefore not considered for the estimation of the index. For the calculation the most actual of the following price types of the option is selected:

- settlement price
- mid-price
- trade price

Example:

Settlement prices are from previous trading day

STRIKE	SETTLE	BID_ TIME	BID	ASK_ TIME	ASK	MID_ TIME	MID	TRADE_ TIME	TRADE	price for calculation
825	76.7		--		--		--		--	76.7
850	53.71		--		--		--	09:05	54.01	54.01
875	37.51	09:04	33.7	09:05	34.4	09:05	34.05			34.05
900	22.54	09:03	17.29	09:05	19.53	09:05	18.41	09:01	20.21	18.41

[37] refers to old market making model at Deutsche Börse

$K_{i,b}$

The sub-indices are calculated according to the following formula:

$$VSTOXX_i = 100 \cdot \sqrt{s_i^2}$$

Equation 25: Volatility index using new method

where:

$$s_i^2 = \frac{2}{T_i} \sum_j \frac{\Delta K_{i,j}}{K_{i,j}^2} \cdot R_i \cdot M(K_{i,j}) - \frac{1}{T_i}\left(\frac{F_i}{K_{i,b}} - 1\right)^2, \ i = 1,2,...,8 \text{ (see appendix)}$$

Equation 26: Implied variance

and:

T_i Time to expiration of the i^{th} index option

F_i Forward index level derived from i^{th} index option prices for which the absolute difference between call and put prices is smallest. Therefore this relation can be expressed as:

$$F_i = StrikePrice_{MinDiff|C-P|} + R_i * min\,|CallPrice - PutPrice|_i$$

(Note: If no unique minimum exists then the average of the forward index levels under consideration is taken as Forward index level)

$K_{i,j}$ Strike price of the j^{th} out-of-the-money option of the i^{th} equity index option in ascending order; a call if $K_{i,j} > F_i$ and a put if $K_{i,j} < F_i$;

$$K_{i,j} < K_{i,j+1} \ \forall j$$

$\Delta K_{i,j}$ Interval between strike prices – half the distance between the strikes on either side of $K_{i,j}$

$$\Delta K_{i,j} = \frac{K_{i,j+1} - K_{1,j-1}}{2}$$

($\Delta K_{i,j}$ is the difference between the lowest strike and the next higher strike for the lowest selected strike and the difference between the highest strike and the next lower strike for the highest selected strike.)
(Note: Ignore a strike if a price is not available)

$K_{i,b}$ First strike below the forward index level F_i; b = # strikes $\leq F_i$

R_i Refinancing Factor of the i^{th} index option

($R_i = e^{r_i T_i}$; r_i as the risk-free interest rate to expiration of the i^{th} index option)

$M(K_{i,j})$	The price for each option with strike $K_{i,j} \neq K_{i,b}$
$M(K_{i,b})$	The put/call average with strike $K_{i,b}$

The sub-indices are calculated up to 2 days remaining to settlement day of expirations. Each new sub-index is disseminated for the first time on the second trading day of the relevant index option.

The time to expiration is given by:

$$T_i = \left(T_{current\ day} + T_{settlement\ day_i} + T_{other\ days}\right) / T_{year}$$

$T_{current\ day}$	= time remaining until midnight of the current day
$T_{settlement\ day_i}$	= time from midnight until 8:30a.m. (configurable calculation time) on settlement day i
$T_{other\ days}$	= time between current day and settlement day
T_{year}	= time in current year

To calculate the time to expiration T for all sub-index expirations (under consideration of day light saving times) one uses seconds as the unit:

$$T_i = \left(N_{current\ day} + N_{settlement\ day_i} + N_{other\ days}\right) / N_{year}$$

$N_{current\ day}$	= seconds remaining until midnight of the current day
$N_{settlement\ day_i}$	= seconds from midnight until configured calculation time on settlement day i
$N_{other\ days}$	= seconds between current day and settlement day
N_{year}	= seconds in fixed calendar year of 365 days

Example: a one month expiry (1M):

Trading day: 2004/04/29, tick time: 10:54:00, settlement time 08:30

$N_{current\ day}$	= ((24-10) * 60 – 54) * 60 = 47,160
$N_{settlement\ day_i}$	= (8 * 60 + 30) * 60 = 30,600
$N_{other\ days}$	= 60 * 60 * 24 * 21 = 1,814,400
$N_{alldays}$	= $N_{current\ day}$ + $N_{settlement\ day_i}$ + $N_{other\ days}$ = 1,892,160

N_{year} = 365 * 24 * 60 *60 = 31,536,000

T_i = $N_{alldays}$s / N_{year} = 0.06

Qualifier	1M	2M	3M	1Q	2Q	3Q	1H	2H
Expiration	2004/05	2004/06	2004/07	2004/09	2004/12	2005/03	2005/06	2005/12
Settlement Date	21.05.2004	18.06.2004	16.07.2004	17.09.2004	17.12.2004	18.03.2005	17.06.2005	16.12.2005
other days	21	49	77	140	231	322	413	595
$N_{other\ days}$	1814400	4233600	6652800	12096000	19958400	27820800	35683200	51408000
$N_{alldays}$	1892160	4311360	6730560	12173760	20036160	27898560	35760960	51485760
T_I	0.06	0.13671	0.21342	0.38602	0.63534	0.88465	1.133972	1.63260

A linear interpolation is used to determine interest rates with maturities matching those of the index option.

$$r_i \equiv r(T_i) = \frac{T_{k+1} - T_i}{T_{k+1} - T_k} r(T_k) + \frac{T_i - T_k}{T_{k+1} - T_k} r(T_{k+1});$$

where $T_k \leq T_i < T_{k+1}$

For all interest rates the time to expiration in seconds has to be calculated and the actual rate has to be gathered.

	EONIA	1M Euribor	2M Euribor	3M Euribor	4M Euribor	5M Euribor	6M Euribor	7M Euribor
days	1	30	60	90	120	150	180	210
N_{days}	86400	2592000	5184000	7776000	10368000	12960000	15552000	18144000
T_I	0.00273	0.08219	0.16438	0.24657	0.32876	0.41095	0.49315	0.57534
rate	2.04	2.056	2.063	2.073	2.082	2.093	2.106	2.123

	8M Euribor	9M Euribor	10M Euribor	11M Euribor	12M Euribor	REX 1Y	REX 2Y
days	240	270	300	330	360	360	720
N_{days}	20736000	23328000	25920000	28512000	31104000	31104000	62208000
T_I	0.65753	0.73972	0.82191	0.90410	0.98630	0.98630	1.97260
rate	2.144	2.165	2.188	2.213	2.239	2.172	2.5344

The interpolation leads to the following interest rates:

Qualifier	1M	2M	3M	1Q	2Q	3Q	1H	2H
$N_{all\ days}$	1892160	4311360	6730560	12173760	20036160	27898560	35760960	51485760
r_i	2.05153	2.06064	2.06896	2.08827	2.12303	2.16792	2.23854	2.39313

The following refinancing factors for the 8 maturities are then calculated:

$$R_i = e^{r_i T_i}.$$

Qualifier	1M	2M	3M	1Q	2Q	3Q	1H	2H
T_i	0.06	0.136712329	0.2134247	0.3860274	0.6353425	0.8846575	1.133972603	1.6326027
r_i	2.05153	2.06064	2.06896	2.08827	2.12303	2.16792	2.23854	2.39313
R_i	1.13098	1.32540	1.55514	2.24042	3.89063	7.04637	12.48486	51.09627

For 1M (1 month expiries) options the absolute smallest difference, forward price F_i and strike price $K_{i,0}$ are determined using the not filtered out "out-of-the-money" calls and puts:

min$|CallPrice - PutPrice|$ = 0.43

Strike Price = 900

1M options			
Strike Price K	**Call**	**Put**	**Absolute Difference**
775	125.48	0.11	125.37
800	100.79	0.41	100.38
825	76.7	1.3	75.4
850	54.01	3.6	50.41
875	34.05	8.64	25.41
900	18.41	17.98	0.43
925	8.07	32.63	24.56
950	2.68	52.23	49.55
975	0.62	75.16	74.54
1000	0.09	99.61	99.52
1025	0.01	124.52	124.51

The out-of-the-money option prices are cut off if they are priced less than Z (which can be parameterized). Z could be for example 0.5 option points.

(1) The forward index level F_i for all options is calculated as follows:

$$F_i = StrikePrice_{MinDiff|C-P|} + R_i * min\,|CallPrice - PutPrice|_i$$

e.g.

R_{1M} = 1.130988311 %

F_{1M} = 900.490

(2) K_0 is the strike price immediately below the forward index levels F_i.

F_{1M} = 900.490

K_0 = 900

If the absolute call-put differences of two or more options with different strike prices are identical, for each of these strike prices, a forward index level has to be calculated.

1M options			
Strike Price K	**Call**	**Put**	**Absolute Difference**
900	30.41	17.98	12.43
925	44.20	32.63	12.43

F_{1M} (900) = 914.06

F_{1M} (925) = 939.06

The average of these forwards is used to determine K_0.

Average (F_{1M} (900) + F_{1M} (925)) = 926.56

K_0 = 925

$M_{i,j}$ Price of the selected j^{th} out-of-the-money option of the i^{th} index option in ascending order;

a call if $K_{i,j} > F_i$ and a put if $K_{i,j} < F_i$; $K_{i,j} < K_{i,j+1}$ $\forall j$

Example:

1M options				
Strike Price	**Call**	**Put**	**$M_{i,j}$**	
775	125.48	0.11	= Put	0.11
800	100.79	0.41	= Put	0.41
825	76.7	1.3	= Put	1.3
850	54.01	3.6	= Put	3.6
875	34.05	8.64	= Put	8.64
900	18.41	17.98	=(Call+Put)/2	18.195
925	8.07	32.63	= Call	8.07
950	2.68	52.23	= Call	2.68
975	0.62	75.16	= Call	0.62
1000	0.09	99.61	= Call	0.09
1025	0.01	124.52	= Call	0.01

The $\Delta K_{i,j}$ per strike is calculated a follows

for all options except those with the lowest and highest strike price as:

$$\Delta K_{i,j} = \frac{K_{i+1} - K_{i-1}}{2}$$

for the lowest strike of the selection:

$$\Delta K_{i,j} = K_{i+1} - K_i$$

for the highest strike of the selection:

$$\Delta K_{i,j} = K_i - K_{i-1}$$

1M options					
Strike Price K(i)	**Delta K_{ij}**		**Call**	**Put**	**$M_{i,j}$**
775	=K(i+1) - K(i)	25	125.48	0.11	0.11
800	=(K(i-1) - K(i+1))/2	25	100.79	0.41	0.41
825	=(K(i-1) - K(i+1))/2	25	76.7	1.3	1.3
850	=(K(i-1) - K(i+1))/2	25	54.01	3.6	3.6
875	=(K(i-1) - K(i+1))/2	25	34.05	8.64	8.64
900	=(K(i-1) - K(i+1))/2	25	18.41	17.98	18.195
925	=(K(i-1) - K(i+1))/2	25	8.07	32.63	8.07
950	=(K(i-1) - K(i+1))/2	25	2.68	52.23	2.68
975	=(K(i-1) - K(i+1))/2	25	0.62	75.16	0.62
1000	=(K(i-1) - K(i+1))/2	25	0.09	99.61	0.09
1025	=K(i) – K(i-1)	25	0.01	124.52	0.01

Implied variance s_i^2 of the index options is then estimated[38] as shown below:

$$s_{\text{var}}^2 = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*} e^{rT} - 1 \right) - \ln\left(\frac{S_*}{S_0} \right) + e^{rT} \int_0^{S_*} \frac{1}{K^2} P(K) dK + e^{rT} \int_{S_*}^{\infty} \frac{1}{K^2} C(K) dK \right)$$

Equation 27: Theoretical Value of Implied Variance

The terms are from left to right:

The financing cost of rebalancing the position in underlying shares

A short position in $1/S_*$ forward contracts struck at S_*

A short position of a logarithmic contract paying $\ln(S_*/S_0)$ at expiration

A long position in $(1/K^2)$ put options with price P struck at K, for a continuum of all out-of-the-money strikes

A long position in $(1/K^2)$ call options with price C struck at K, for a continuum of all out-of-the-money strikes.

In its discrete form implied variance can be represents as follow:

$$s_i^2 = \frac{2}{T_i} \sum_j \frac{\Delta K_{i,j}}{K_{i,j}^2} \cdot R_i \cdot M_{i,j} - \frac{1}{T_i}\left(\frac{F_i}{K_{i,0}} - 1 \right)^2$$

Equation 28: Discrete Formula of Implied Variance

[38] See mathematical appendix for analytical derivation of new methodology.

1M options					
Strike Price K(i)	ΔK	Call	Put	$M_{i,j}$	$\frac{\Delta K_{i,j}}{K_{i,j}^2} R_i M_{ij}$
775	25	125.48	0.11	0.11	0.000004581
800	25	100.79	0.41	0.41	0.000016023
825	25	76.7	1.3	1.3	0.000047773
850	25	54.01	3.6	3.6	0.000124627
875	25	34.05	8.64	8.64	0.000282257
900	25	18.41	17.98	18.195	0.000561842
925	25	8.07	32.63	8.07	0.000235905
950	25	2.68	52.23	2.68	0.000074274
975	25	0.62	75.16	0.62	0.000016313
1000	25	0.09	99.61	0.09	0.000002251
1025	25	0.01	124.52	0.01	0.000000238

$$\sum_i \frac{\Delta K_{ij}}{K_{ij}^2} R_i M_{ij} = 0.06648277$$

$$\sigma_{1M}^2 = 0.0664772$$

Then the sub-index $VSTOXX_i$ is calculated as follows:

$$VSTOXX_i = 100 \cdot \sqrt{s_i^2}$$

Example:

$$VSTOXX_{1M} = 100 \cdot \sqrt{0.0664772} = 25.7832$$

Apart from the sub-indices for the various individual maturities, the main volatility index is determined using a constant remaining time to expiration of 30 days (this index is not linked to a specific time to expiration). It is calculated in the same way as the old methodology. The main index is determined by linear interpolation of the sub-indices which are nearest to a remaining time to expiration of 30 days. In this case, the two nearest available indices are used, which are as close to the time to expiration of 30 calendar days as possible.

Therefore the main $VSTOXX$ index level is the result of a linear interpolation between $VSTOXX_i$ and $VSTOXX_{i+1}$ which encloses or boundaries the remaining lifetime of 30 days:

$$VSTOXX = \sqrt{\left[T_i \cdot VSTOXX_i^2 \cdot \left[\frac{N_{i+1} - N_{fixed}}{N_{i+1} - N_i} \right] + T_{i+1} \cdot VSTOXX_{i+1}^2 \cdot \left[\frac{N_{fixed} - N_i}{N_{i+1} - N_i} \right] \right] * \frac{N_{365}}{N_{fixed}}}$$

Equation 29: Linear Interpolation of volatility sub-indices

N_{fixed} = 30 days. Fixed remaining lifetime of main index

N_{365} = time for a standard year = 365 * 24 * 3600 = 31536000

Example:

i = 1M

$i+1$ = 2M

$VSTOXX_{1M}$ = 25.7832

$VSTOXX_{2M}$ = 25.2326

$VSTOXX =$

$$\sqrt{\left[0.0598361 \cdot 25.7832^2 \cdot \left[\frac{4311360 - 3888000}{4311360 - 1892160} \right] + 0.1363388 \cdot 25.2326^2 \cdot \left[\frac{3888000 - 1892160}{4311360 - 1892160} \right] \right] * \frac{31536000}{3888000}}$$

= 25.2454

3.3 Improvements Incorporated into the New Methodology

Now there are some improvements in the new methodology which are worthy of being mentioned. The new methodology not only uses the four nearby options to the at-the-money point, but also utilizes a range of out-of-the money options around this point, thus covering and capturing more of the volatility surface (volatility skew) than the old methodology. This therefore makes the main volatility index less sensitive to individual options. The use of more options within the calculation and the avoidance of an option pricing model in the estimation make the calculated volatility value a much better estimation of the market participants’ true anticipation of volatility.

Furthermore, the volatility index calculated using the new methodology is easier to hedge. Implied variance calculated using this methodology can be hedged using a static strip of options. But hedging the square root of implied variance, one would require to dynamically hedging the strip of options, but this type from hedging is effectively much more costly.

However one should note that implied variance isn't implied volatility. By considering the relationship between the square root of implied variance and implied at-the-money volatility, one can see strong similarities, but the two concepts are not completely identical. While the at-the-money implied volatility (which is derived from the old methodology) contains information about at-the-money option prices only, the square root of implied variance (using the new methodology) contains information about the entire volatility skew, just like implied variance itself. Despite the different ways of expressing volatility, implied variance and implied volatility bear strong similarities in their usage for trading and hedging.

As mentioned above, using the square root of implied variance prevents the possibility of hedging futures contracts on such an index statically. However, the fact that the square root of implied variance includes data for the entire volatility skew means that this provides added benefits for dynamic hedging methods in the form of significantly higher stability compared to using implied at-the-money volatility. Market makers are thus sophisticated enough to hedge products based on the square root of implied variance as an underlying.

Another improvement worthy of mentioning is the fact that the new methodology measures expected volatility as financial theorists, risk managers, and volatility traders have come to measure it. As a result, the new methodology more closely conforms to financial and risk industry practices. It is simpler, yet it yields a more robust measure of expected volatility, due to its covering of the volatility skew.

4. The Historical Time Series of the Family of the Volatility Indices

In order to analyze the properties of the volatility index calculated using the new methodology, a time series analysis was executed. This was very important to ensure that before launch of the new indices all expectations regarding their properties (like anti-correlation between volatility index and equity index) were met by the new

calculation method. It was also an ideal opportunity to be able to test different constellations of the parameters which were to be used in the productive system.
The time series were calculated using a statistical, mathematical language and environment tool called R©. The source code can be seen in the mathematical appendix below. This enabled the acceleration of calculation of the time series of the index levels using different constellations of the parameters to be implemented.

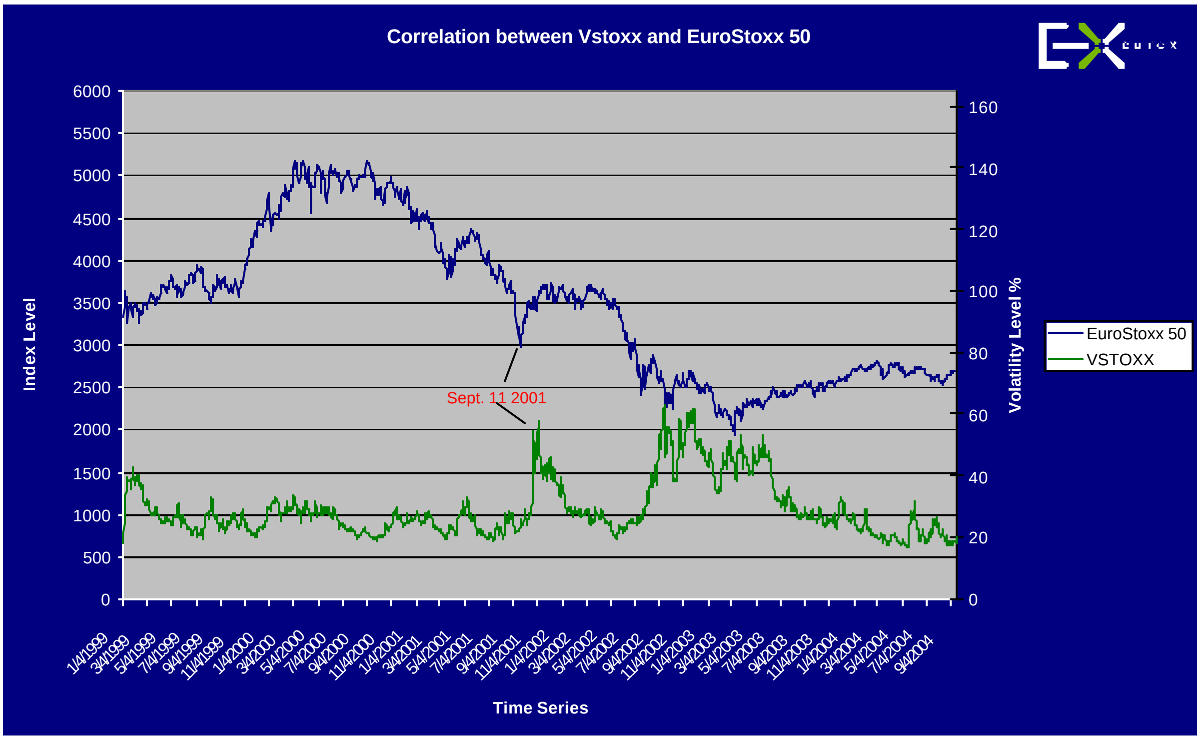
4.1 Data Source of the Historical Time Series

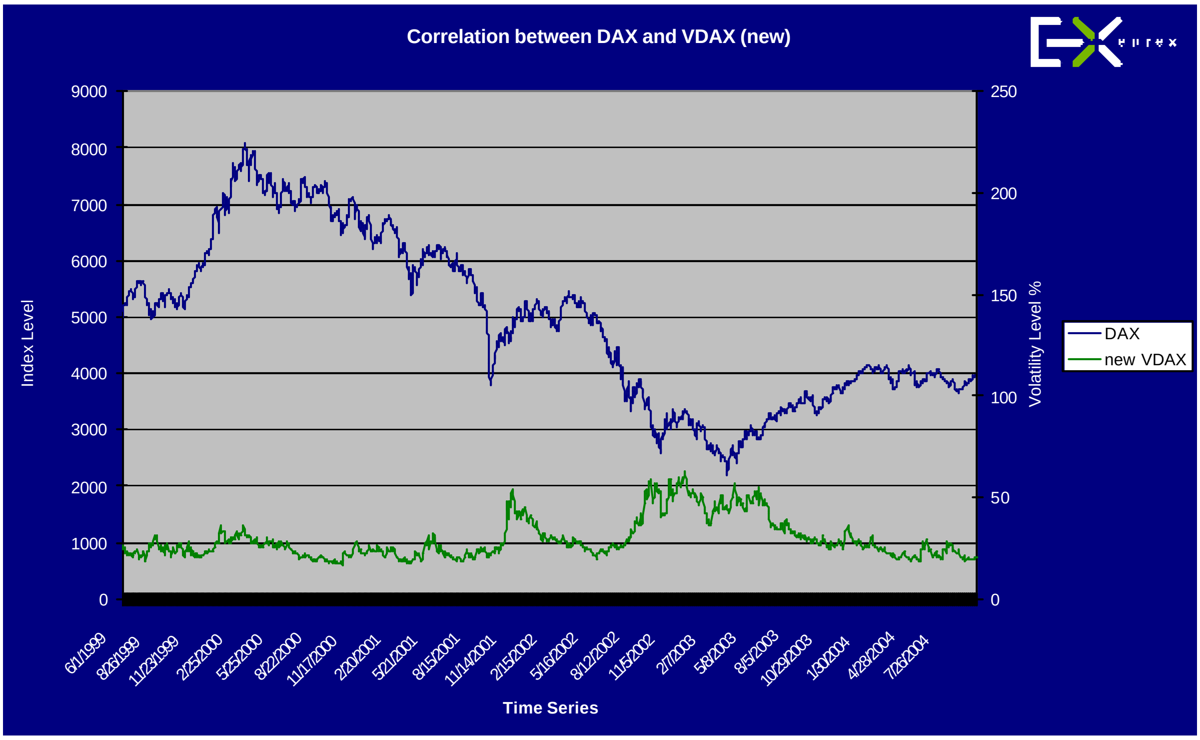
Time series calculations were executed for three equity indices. These were for the three new volatility indices of the Deutsche Börse AG which will be launched in the second quarter of 2005. They are namely, the VSTOXX, the volatility index on EURO STOXX 50 options traded at the Deutsche Börse AG; the new VDAX which is a volatility index on the DAX options traded at the Eurex; and finally the VSMI, the volatility index on the Swiss Exchange's equity index SMI options also traded at the Eurex.
The EURO STOXX 50 index provides a blue chip representation of sector leaders within the Eurozone which includes industry leaders in countries like Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal and Spain. The unique aspect of this index is that it captures approximately 60% of the free-float market capitalization of the Dow Jones EURO STOXX Index, which in turn covers approximately 95% of the free-float market capitalization of the represented Eurozone countries. The option contract on the EURO STOXX 50 is a contract with a value 10 EUR pro EURO STOXX 50 index point. There is a minimum price change of 0.1 point which corresponds to 1 EUR.
The expiration day is always the third Friday of the expiration month at 12:00 CET, as long as this day is a trading day. Otherwise it is the first trading day before. The series used in the calculation of the implied volatility were the one, two and three months to expiration, the three following quarter in chronological order, and finally the following two half-year maturities. These are also the same expirations used for the VDAX and VSMI implied volatility estimations.
The DAX® comprises the 30 largest German shares with the highest turnover, representing roughly 70 per cent of the overall market capitalization of domestic listed companies. The trading in these shares accounts for more than 80 per cent of Germany's exchange-traded equity volumes. Based on its real-time concept, with

recalculations carried out every 15 seconds, the DAX® provides a comprehensive and up-to-date tracking image of the German stock market.
The option contract on this index is one of the products at Eurex® with the highest trading volume, and ranks as one of the top index options contracts worldwide. The option contract's specifications are almost identical to those of the EURO STOXX 50 option with the exceptions that the value per DAX index point is 5 EUR and the minimum point change has a value of 0.50 EUR. The expiration day is always the third Friday of the expiration month at 13:00 CET, as long as this day is a trading day. Otherwise it is the first trading day before. The VDAX (old) and VDAX (new) are calculated each on the basis of eight expiry months with a maximum time to expiration of two years as discussed above.
The SMI (Swiss Market Index) is Switzerland's blue-chip index, which makes it the most important market indicator for the country. It is made up of a maximum of 30 of the largest and most liquid Swiss price index's large- and mid-cap equities. The securities contained in the SMI currently represent more than 90 % of the entire market capitalization, as well as of 90 % trading volume, of all Swiss equities listed on the Swiss Exchange. Because the SMI is considered to be a mirror of the overall Swiss stock market, it is used as the underlying index for numerous derivative financial instruments such as options, futures and index funds. The option traded at Deutsche Börse on the SMI is called the OSMI and it is the underlying for the VSMI volatility index. It's very similar in structure to both options on the EUROSTOXX 50 and DAX respectively. The SMI index point is 5 CHF and the minimum point change of 0.1 has a value of 1 CHF. The expiration day is always the third Friday of the expiration month at 17:20 CET, as long as this day is a trading day. Otherwise it is the first trading day before.
In order to calculate each volatility index the daily settlement prices (i.e. of all qualified calls and puts of the options on the indices traded at the Eurex) were used as the data source. The interest rates used in the calculation were the EURIBOR, LIBOR, REX and the Swiss Banking Association's Interest Rates.

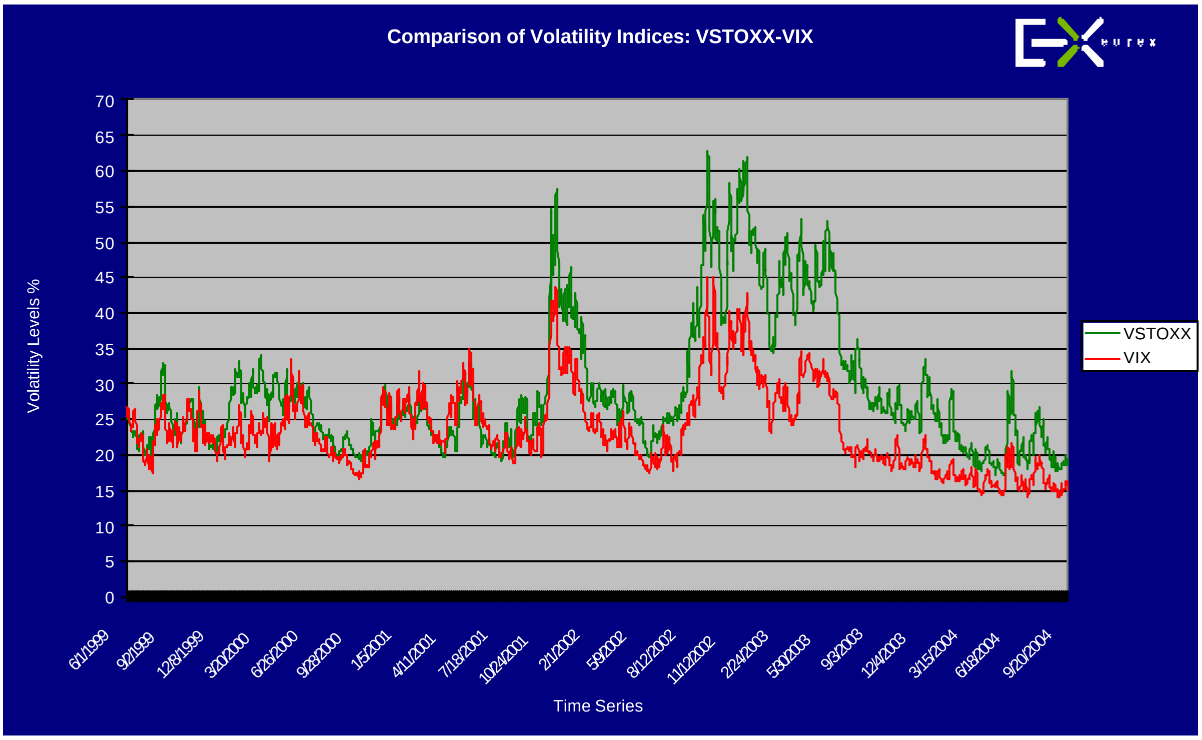
4.2 Analysis of Historical Time Series

Evaluating the results of the implied volatility time series calculation, below clearly indicates that the VSTOXX and VDAX are negatively correlated to their corresponding underlying, thus confirming them as indicators of market participant's pessimistic sentiments. One see that the volatility index rises when the equity index falls but not always equally proportional.



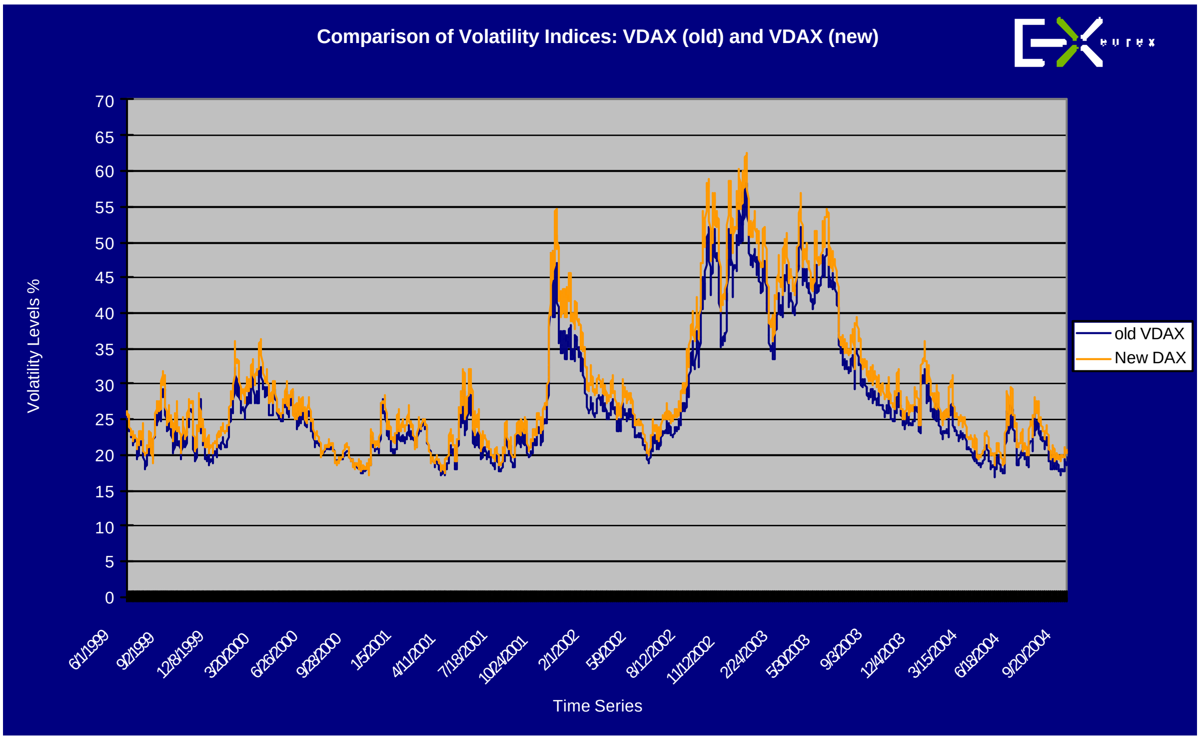


In comparison with the volatility index VIX on the S&P 500 index at the Chicago Board of Exchange, which as a matter of fact follows a methodology closely to that of the new methodology; one can observe a similar trend in the movement of volatility:

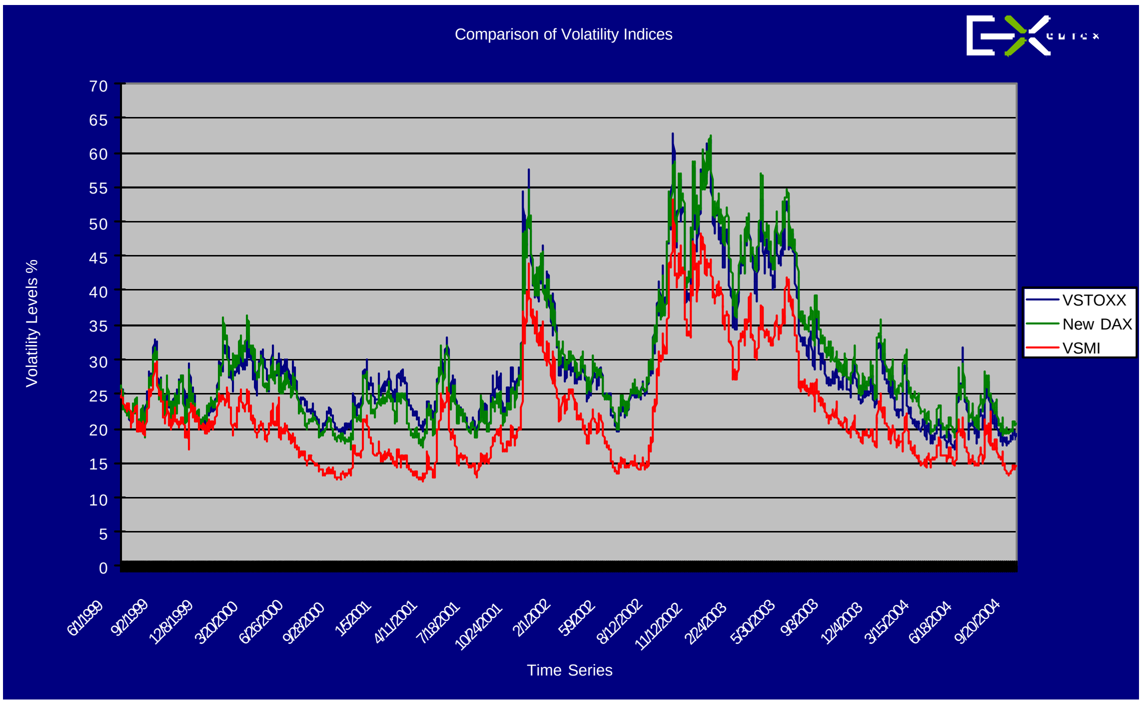


Therefore investors familiar with the VIX can get used to the VSTOXX without having to relearn or rethink. Due to the spread between the two volatility indices, the launch of the VSTOXX will open up new possibilities for arbitrage trading between the US and the European volatility levels. The VSTOXX runs typically higher than the VIX. This is based on the fact that the VIX is derived from the S&P 500 index options representing the broad and deep US market. European markets are much less deep and therefore more volatile. In addition, the EuroStoxx 50 consists of 10 times fewer constituents than the S&P 500.

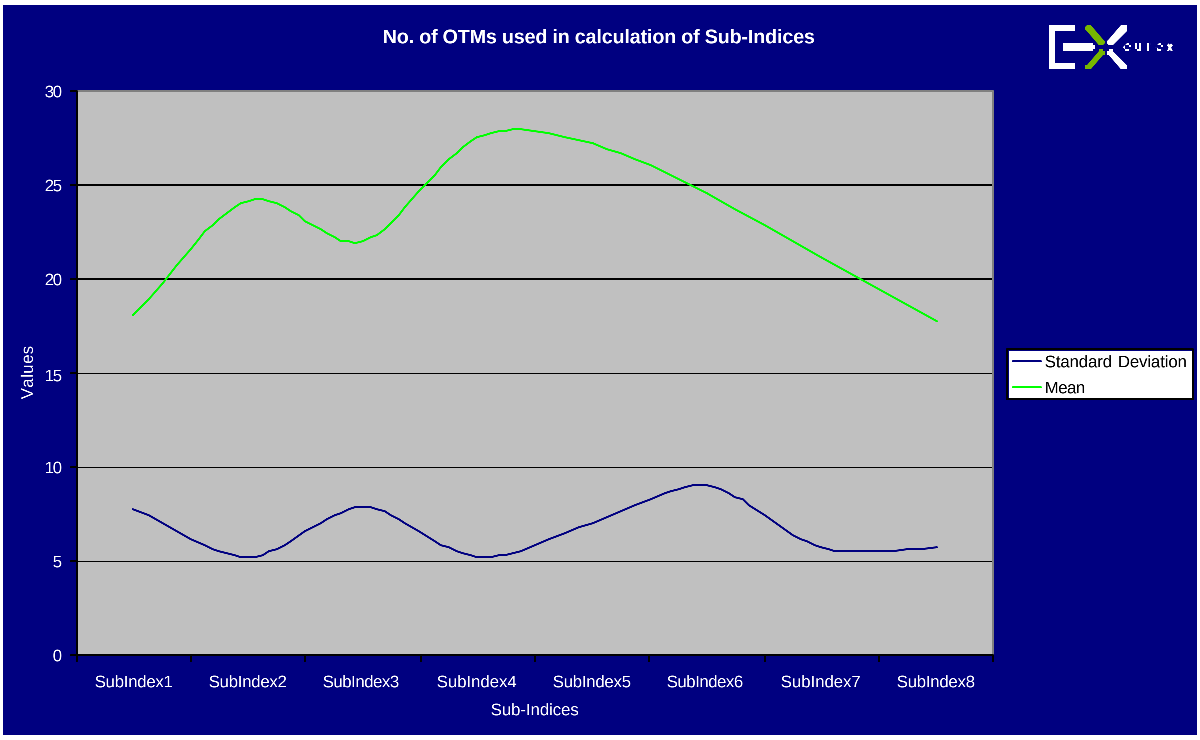
A comparison of the times series of the VDAX (volatility index on the DAX) calculated using old and new methodologies exhibits only minute differences. This can be seen as an added advantage for institutions which have already set in place warrants and certificates on the old VDAX. They can continue to use their present hedging strategies on the new VDAX. The old VDAX is typically lower that the new VDAX mainly because the old VDAX is calculated on a rolling fixed 45 days to expiration versus a rolling fixed 30 days to expiration of the new VDAX calculation. This is due to the fact that the old VDAX is further down the term structure which is most of the time downward sloping.

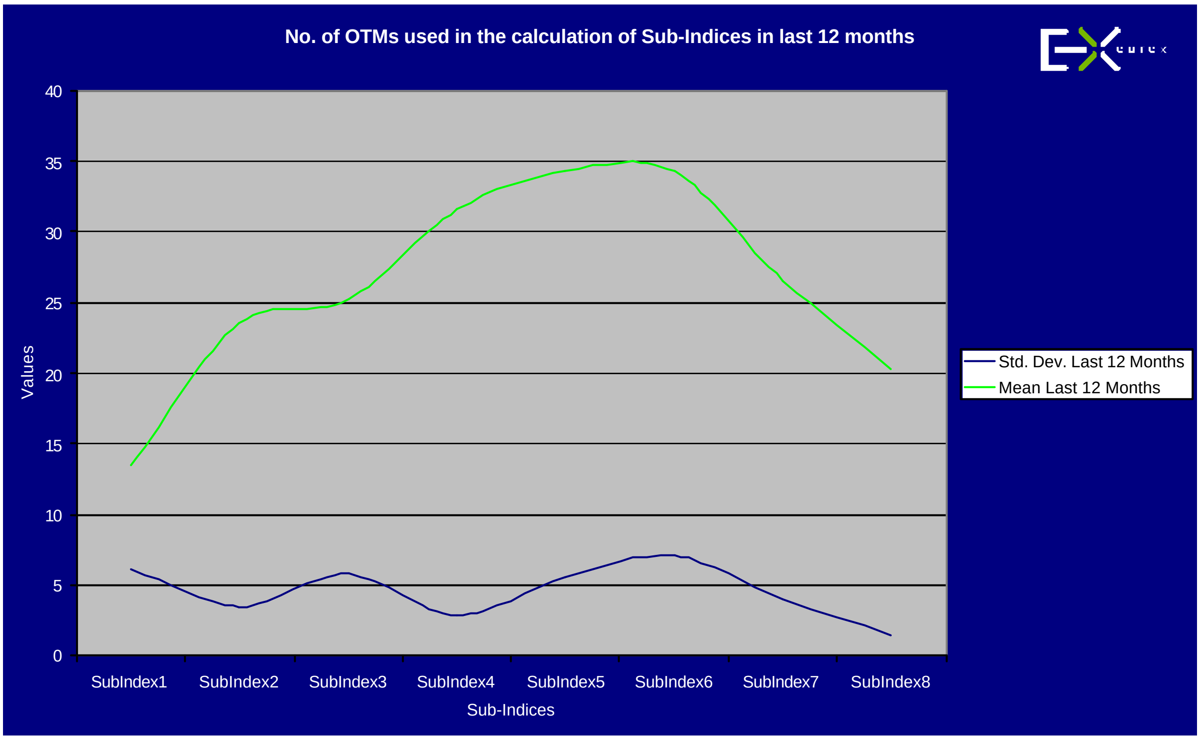


Below, the time series of all calculated indices are illustrated. One can observe that the VSMI runs slightly low than the VSTOXX and VDAX. This is mainly due to some unique characteristics of the Swiss economy such as its low interest rates and relatively stable market.

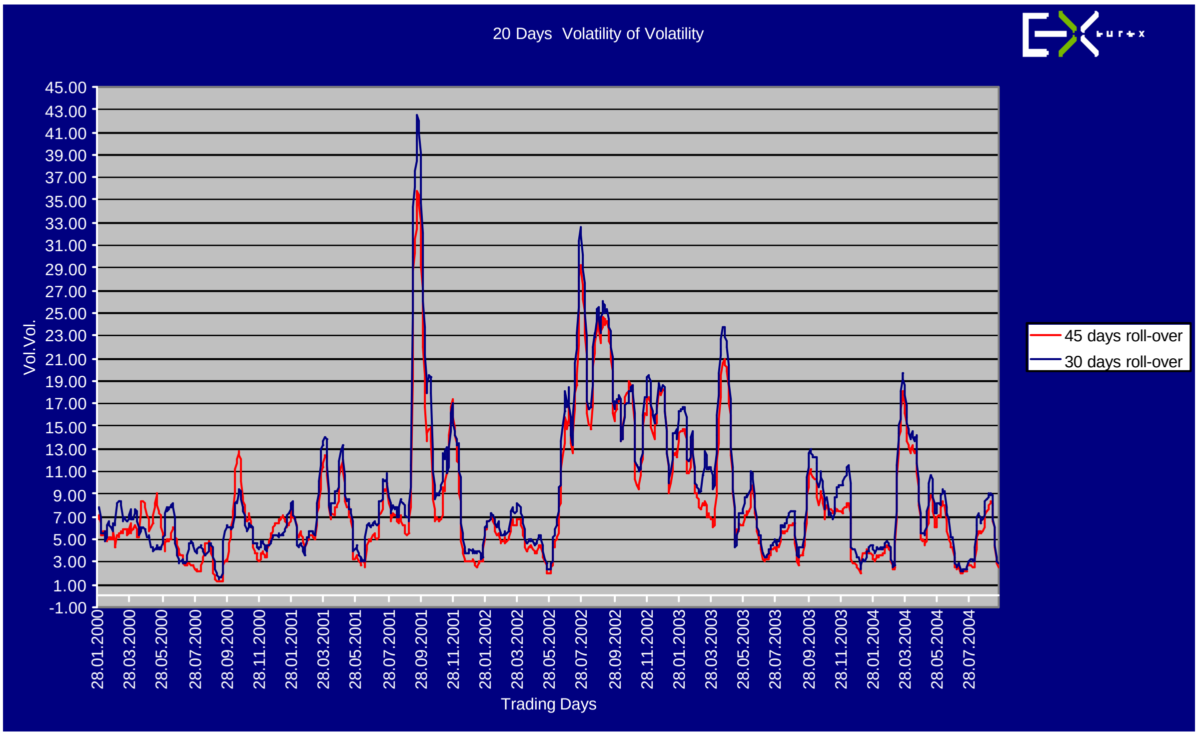


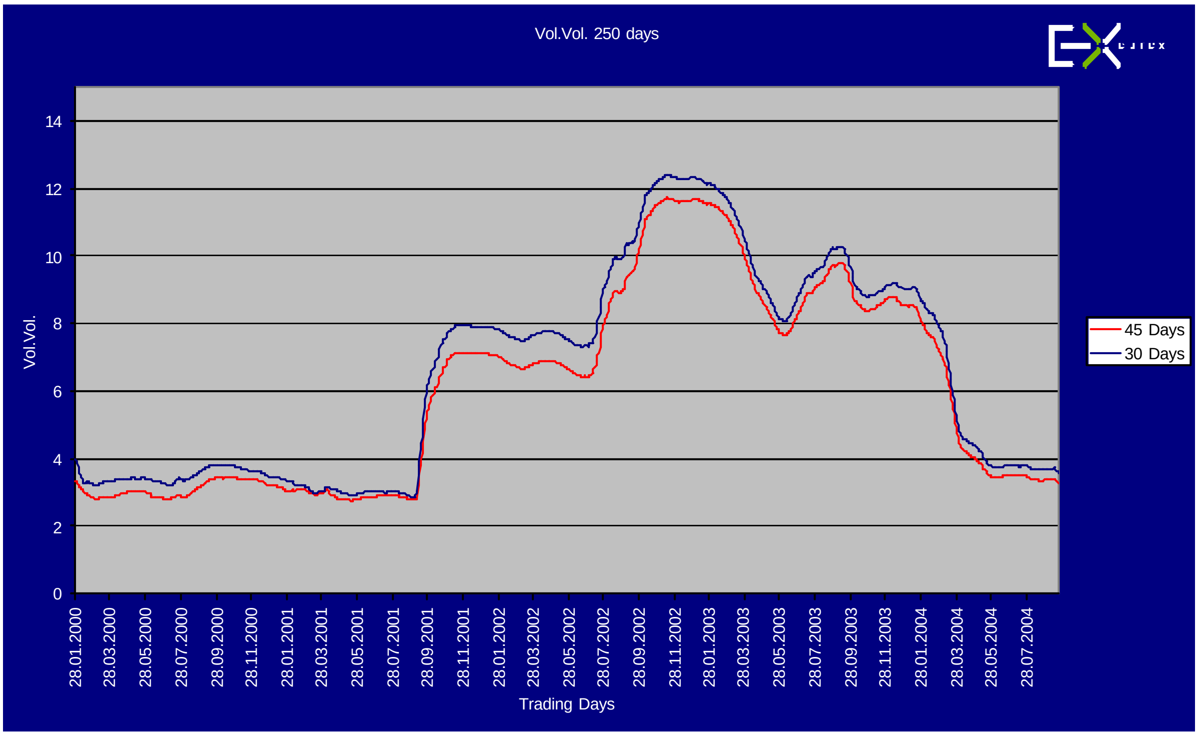
When implementing dynamic hedging strategies for a derivative product on the volatility index, the decision on how many strikes will be used for the hedging process is very crucial for the effectiveness of the strategy and the level of the transaction cost to implement such a strategy. A hedging strategy which utilizes a small number of strikes may prove to be ineffective because, only a small area of the volatility surface is considered. This means that the strategy does not fully replicate total market consensus of volatility in its calculations. On the other hand, using too many strikes (i.e. incorporating more strikes on the volatility surface) in the strategy can increase the transaction cost of executing such a strategy to such a high level that is becomes unprofitable to implement. The strategy may then be proven to be inefficient. Financial institutions and brokerage firms normally use about 10 strikes to hedge volatility derivatives based on the square root of implied variance. The calculation of the volatility index using the new methodology shows that the means along all sub-indices range between 15 and 35 out-of-the-money (OTM) strikes when calculating volatility. This ensures the efficiencies of the new methodology in estimating volatility and reiterates that it uses a sufficient number of strikes for hedging purposes.



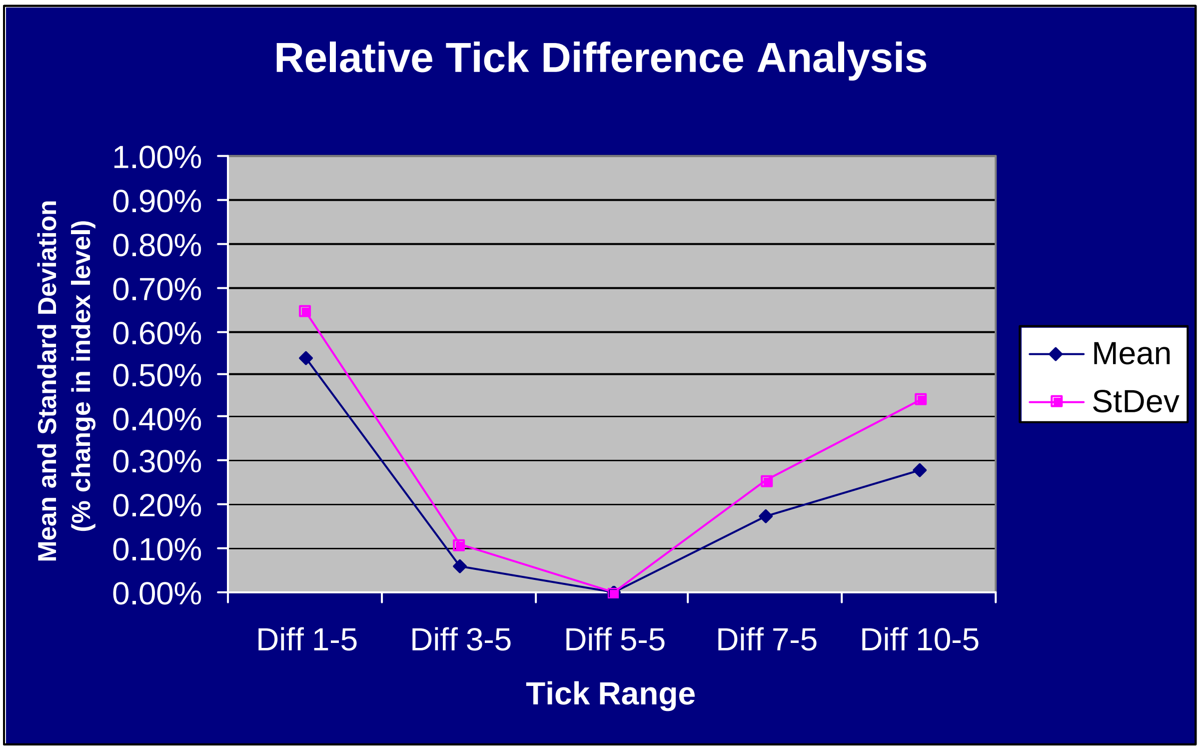


The magnitude, at which volatility alters, can be expressed as the volatility of volatility "Vol.Vol." (i.e. the standard deviation of volatility). It has been empirically observed that the closer an option is to its expiration date, the more volatile is its price. Therefore one would expect that if the rolling fixed time to expiration used for calculating the volatility index were calculated using a 45 days fixed time to expiration and a 30 days fixed time to expiration, the time series calculated with the 30 days fixed time to expiration would show a higher volatility of volatility than that of the 45 days fixed time to expiration time series. This is an important aspect of volatility which may prove viable for intra-day arbitrage traders of volatility. This simply means that they would be able to lock in higher bid-ask spreads using the volatility index calculated on a 30 days fixed time to expiration than that of a volatility index calculated using a 45 days fixed time to expiration.





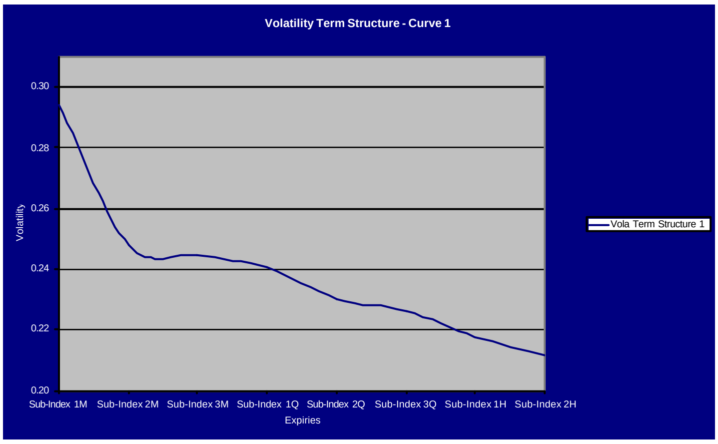
Due to the fact that the cut-off point of the minimal tick is set at 0.5, it was necessary to evaluate the consequence of such a decision. Therefore a relative tick difference analysis between various cut-off points (0.1, 0.3, 0.5 , 0.7, 1 bases points) was executed. The results below illustrate that the mean and standard deviation of such an action (i.e. loss in contribution) to the value of volatility is very small.

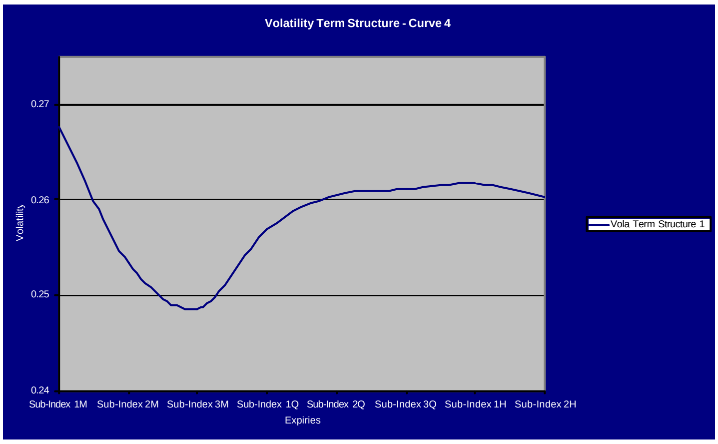


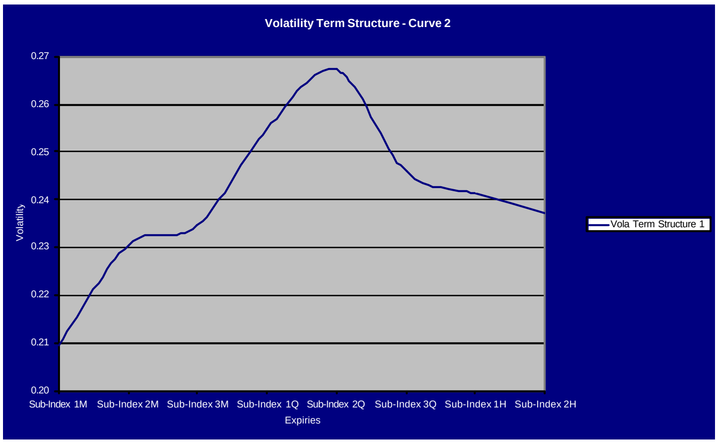
Normally traders aren't interested only in static volatility surfaces. They also want to know how the volatility skew will respond to the passage of time and a change in the underlying’s value. Therefore, an analysis of the term structures of volatility produced by the volatility index calculation, are necessary.

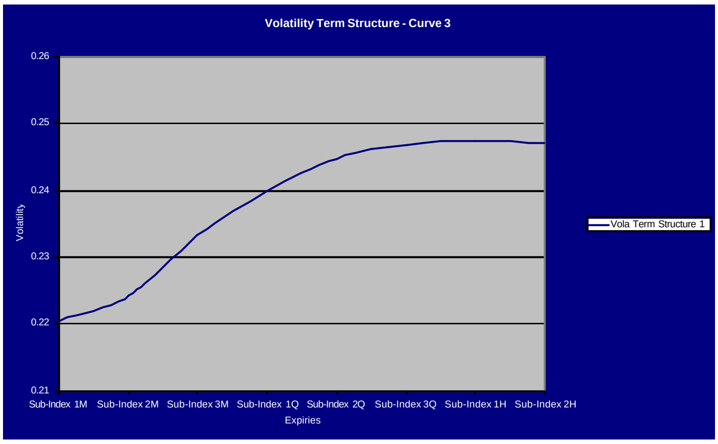
The volatility term structure (VTS) reflects market expectations of asset volatility over different horizons. These expectations change over time, giving a dynamic structure to the VTS. As shown below, this structure readily changes over time. This means that the volatility skew complicates the tasks of pricing and hedging options. Changes in implied volatilities that are expected to accompany changes in the value of the underlying over time will impact the option's value.

Looking at the curves of different volatility term structures one can observe how the term structure of volatility (in this case the VSTOXX volatility term structures) changes over time when the value of the underlying (EUROSTOXX 50) alters. If one assumes that the value of the EUROSTOXX 50 index is on the rise, then the volatility tends to fall and VTS moves from curve 1 to curve 2, i.e. the short end is more sensitive to changes and therefore reacts faster than the long end of the VTS. After a while the value of the underlying begins to stabilize resulting in a shift of the VTS from curve 2 to curve 3. Finally, when the EUROSTOXX 50 index value is on a decline, i.e. volatility is on a rise, the VTS shifts from curve 3 to curve 4.









5. **Conclusion**

This thesis has introduced one of the most important measures of risk in modern day finance. Due to financial market crises of the past, volatility trading has grown to unprecedented magnitude. As a result, many trading strategies and concepts have evolved which have as their main focus, hedging against market volatility risk. There has also been a steady rise in the number financial houses, which only trade in volatility. Volatility is sometimes vaguely used interchangeably with sample standard deviation. Although similar, there is a unique difference between the two. While volatility normally assumes a relation to a standard distribution like that of the normal distribution function, sample standard deviation does not possess such an association.

Volatility can be measured in two ways. Implied volatility which is forward looking, reflects the volatility of the underlying asset given its markets option price. The other way is that of historical volatility which is backward looking. This type of volatility is derived by estimating volatility using historical market data.

Three commonly used methods of estimating historical volatility are the Close-Close, the High-Low and the High-Low-Open-Close volatility estimators. The last aforementioned volatility estimator is considered to be the most efficient of the three, due the fact that it incorporates more market information into it calculation than the other two.

Calculation techniques for estimating implied volatility has multiplied dramatically over the last three decades. Using the Black-Scholes Option Pricing Model, one can analytically extract a local implied volatility from the option price through the use of an approximation process. However, due to the fact that a volatility skew has been observed empirically, the assumption of constant volatility over strikes and maturities have proven to be a handicap in replicating this observation.

Innovations in volatility estimation, such as stochastic volatility and auto-regressive conditional heteroskedasticity have tried to integrate well known observed properties of volatilities like skewness, volatility clustering and mean reversion into models. As a result of this trend, metamorphoses of the Black-Scholes model and new concepts have appeared on the modeling landscape, some more noble than their counterparts.

The volatility models which assume stochastic volatility have gain in importance due to the fact that their ability to replicate stochastic volatility enables them to reproduce many properties of volatility which are quite evident in empirical data, most notably fat-tailed distribution and skewness.

ARCH and in particularly the GARCH models are models which include past volatilities into their estimations of future volatilities. It takes into account excess kurtosis (fat-tailed distribution) and volatility clustering which are two important properties of real market volatility observations. This family of models also assumes that the means are conditional, i.e. dependent on observations of the immediate past and additionally that they are autoregressive, meaning mean reverting.

Although these recent innovations in volatility forecasting were catalysts for improvements in the estimation process, they have proven to complicate model building as a whole. For instance new parameters have to be calculated. However, studies like that of Poon and Granger have shown that implied volatility estimators performed better than historical and GARCH type estimators.

Presently volatility trading is mostly executed on the OTC markets in the form of volatility and variance swaps. They are practically traded by three genre, namely the directional traders, the spread traders and the volatility hedgers.

Another classical form of trading volatility is through the use of a Straddle. This entails the purchasing of both call and put options on the same strike.

The advent of volatility indices creates new possibilities to trade pure volatility, i.e. trading in a derivative whose underlying is purely exposed to volatility. At the Deutsche Termin Börse, which is the predecessor of the Eurex, the VDAX was on of the first of such volatility indices and the VOLAX was the first derivative to be based on a volatility index. The old methodology used to calculate VDAX is based on modified Black-Scholes Option Pricing model. This paper introduced the new methodology which will be used in the future to calculate volatility. This new methodology is not based on an option pricing model. Instead it calculates volatility using solely market data derived from the option prices traded in the underlying equity index. Where as the old methodology calculates implied volatility of an ATM point using an approximation process on the Black-Scholes model, the new methodology calculates volatility, implied out of the market, using several selected OTMs calls and puts along with an ATM point, thus including more of the volatility surface in its calculation process. Since no option pricing model is used, the volatility estimation is purely based on market participants' behavior stored in the selected strikes of the options on the equity index used as the underlying. Using the new methodology, the calculation of the volatility index is executed in two distinctive steps. Firstly, eight sub-indices based on the first eight maturities up to two years on the equity index options are calculated and then secondly, the two nearest sub-indices to the rolling fixed 30 days time to expiration are selected for the interpolation process which results in a main volatility index being derived. The inclusion of more options into the calculation process enables the main volatility index to be less sensitive to individual options. Furthermore, the volatility index calculated using the new methodology is easier to hedge when its square, the implied variance, is used. In addition to it's simplicity and market near, the new methodology more closely resembles the volatility measure used by financial and risk practitioners. It's simpler but yet it yields a more robust measure of expected volatility due to its covering of the volatility surface.

The analysis of the historical data of the volatility indices confirmed that the volatility indices do possess an anti-correlation to their respective equity indices. It also confirmed that the old and new methodologies possess very similar paths of volatility over the empirical period of the time series. This makes it easier for users of the old VDAX to adapt to the new methodology.

With the introduction of the new VDAX, VSTOXX and VSMI during the second quarter of 2005, new and innovative ways of trading in pure volatility will be made possible. Futures on the volatility indices are already in plan at the Deutsche Börse AG.

6. Bibliography

Andersen,T., T.Bollerslev and S.Lange,1999,"Forecasting financial market volatility: Sample frequency vis-à-vis forecast horizon", Journal of Empirical Finance, 6, 5, 457-477.

Black,F. and M.Scholes,1973,"The pricing of options and corporate liabilities", Journal of Political Economy 81, 637-659.

Bollerslew,T.,1986,Generalized Autoregressive Conditional Heteroskedasticity", Journal of Econometrics, Vol. 3, 307-327.

Brenner, M. and D. Galai, 1989, „New Financial Instruments for Hedging Changes in Volatility", Financial Analyst Journal, July/August, 61-65.

Carr,P. and D. Madan, 1998 "Towards a Theory of Volatility Trading", Volatility: New Estimation Techniques for Pricing Derivatives, R. Jarrow editor, Risk Books, London, 417-427.

Cootner, P., ed. 1964, "The Random Character of Stock Prices", Cambridge Mass., MIT Press.

Derman, E., and Kani,I., Riding on a smile, Risk, 7 (1994), p. 32--39

Duan, J.C.,1990, "The GARCH Option Pricing Model, unpublished manuscript, McGill University.

Dupire, B., 1994 "Pricing with a Smile". Risk Magazine, 7 18-20.

Engle,R. 1982, "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, Econometrica 52, 289-311.

Figlewski, S.,1997, "Forecasting volatility", Financial Markets, Institutions and Instruments, New York University Salomon Center, 6,1,1-88.

Fleming J., 1998, “The Quality of Markets Forecasts Implied by S&P100 Option Prices”, Journal of Empirical Finance, 5, 317-345.

Garman M.B., M.J. Klass, 1980, “On the Estimation of Security Price Volatility from Historical Data”, Journal of Business, 1980, Vol. 53 (No. 1), 67-78.

Granger, C.W.R., Z. Ding and S. Spear, 2000, “Stylized facts on the temporal and distributional properties of absolute returns, working paper, University of California, San Diego.

Harvey, C.R. and R.E. Whaley, 1992, “Dividends and S&P 100 index option valuation”, Journal of Futures Markets 12(2), 123-137.

Heston, S.L., 1993, “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.”, The Review of Financial Studies, Volume 6, Issue 2, 327-343.

Li, K., D. Weinbaum, 2000, “The Empirical Performance of Alternative Extreme Value Volatility Estimators”, Working Paper, Stern School of Business, New York.

Parkinson M.,1980, “The Extreme Value Method for Estimating the Variance of the Rate of Return”, Journal of Business, 1980, Volume 53 (No. 1), 61-65.

Poon, S.H. and C. Granger, 2002, “Forecasting Volatility in Financial Markets: A Review”.

7. Programming Appendix

Code for Calculating the Historical Time Series of Volatility Indices with R Statistical Programming

START of Codes

```
####################################################################################
### global parameters ########################################################
################################################################################

dataDir              =      "C:/Vola_Data_OESX"
currUnderlying       =      "ODAX"
currRates            =      "euribor"
decimalFactor        =      100
numSubIndex          =      8
DaysInYear           =      365
SecondsInYear        =      365*24*60*60
DiffCloseExpiry      =      4.5
TickFactor           =      5
TickSize             =      0.1
FixedLifetime        =      30
IndexFile            =    "DAX"
Hour_sec             =      3600
Days_Sec             =      86400
Year_Sec             =      31536000
Days_Maturities      =      c(1,7,30,60,90,120,150,180,210,240,270,300,330,360,720)
Int_T_Exp            =      Days_Maturities/DaysInYear


####################################################################################
### master loop over all dates for which we have data ##############################'
################################################################################

for (currYear in 1999:2004)
{

for (currMonth in 1:12)
{

for (currDay in 1:31)
{

       myFile  =        paste(dataDir,"/",currYear,"/",currUnderlying,"_",currYear*100+currMonth,
                               ".txt",sep="")
       if (!file.exists(myFile)) next

       myData =         read.table(myFile,header=T,blank.lines.skip=T,as.is=T)
       myData =
       myData[dimnames(myData)[[1]][myData[,"A_DAY"]==currDay],c(5:ncol(myData))]
       if ( nrow(myData)==0 ) next #i.e. not a trading day
       print(paste("Working on: ",currYear*10000+currMonth*100+currDay,sep=""))
```

```
###############################################################################
#################### Next Step: aligning Calls and Puts ##########################
###############################################################################

	myTmp =
	cbind(myData[dimnames(myData)[[1]][myData[,"A_CALL_PUT_FLAG"]=="C"],
				c("A_EXP_YEAR","A_EXP_MONTH","A_EXERCISE_PRICE")],
				rep(NA,nrow(myData)/2),rep(NA,nrow(myData)/2))
for (i in 1:nrow(myTmp))
	{
		myTmp[i,4] = myData[(
			(myData[,"A_EXERCISE_PRICE"] %in% myTmp[i,"A_EXERCISE_PRICE"]) &
			(myData[,"A_EXP_MONTH"] %in% myTmp[i,"A_EXP_MONTH"]) &
			(myData[,"A_EXP_YEAR"] %in% myTmp[i,"A_EXP_YEAR"]) &
			(myData[,"A_CALL_PUT_FLAG"] %in% "C") ),
			"A_SETTLEMENT_PRICE"]/decimalFactor
		myTmp[i,5] = myData[(
			(myData[,"A_EXERCISE_PRICE"] %in% myTmp[i,"A_EXERCISE_PRICE"]) &
			(myData[,"A_EXP_MONTH"] %in% myTmp[i,"A_EXP_MONTH"]) &
			(myData[,"A_EXP_YEAR"] %in% myTmp[i,"A_EXP_YEAR"]) &
			(myData[,"A_CALL_PUT_FLAG"] %in% "P") ),
			"A_SETTLEMENT_PRICE"]/decimalFactor
	}
	myData =	myTmp
	dimnames(myData)	=	list(seq(1:nrow(myData)),
					c("Exp_Year","Exp_Month","Strike","CPx","PPx"))


####################################################################################
###############
############### Next Step: Creating Lists containing the data of the first 9 expiries ##########
####################################################################################
###############

expList		=	list(NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL)
numExp			=	length(unique(myData[,"Exp_Year"]*100+myData[,"Exp_Month"]))
expVec		=
	sort(unique(myData[,"Exp_Year"]*100+myData[,"Exp_Month"]))[1:numSubIndex]



###############################################################################
####### Next Step: Selecting the 8 nearest valid maturities from expList and expMat #########
###############################################################################

names(expList)	=	expVec
expMat		=	cbind(floor(expVec/100),expVec-100*floor(expVec/100),
				rep(NA,length(expVec)))


dimnames(expMat)=	list(expVec,c("Exp_Year","Exp_Month","Exp_Day"))
allExpiries	=	read.table(paste(dataDir,"/Expiration_Dates.txt",sep=""),
				header=T,blank.lines.skip=T,as.is=T)
for ( e in 1:nrow(expMat))
{
	expMat[e,"Exp_Day"]	=	allExpiries[(
					(allExpiries[,"A_YEAR"] %in% expMat[e,"Exp_Year"]) &
					(allExpiries[,"A_MONTH"] %in% expMat[e,"Exp_Month"])
					),"A_DAY"]
	expList[[e]]		=	myData[(
					(myData[,"Exp_Year"] %in% expMat[e,"Exp_Year"]) &
```

```
                                        (myData[,"Exp_Month"] %in% expMat[e,"Exp_Month"])
                                        ),c("Strike","CPx","PPx")]
}

################################################################################
#########Selecting relevant interest rates that match the current date#####################
################################################################################

i_euribor_current=paste(currDay,currMonth,currYear,sep = ".")

for(i in currYear)
        {
          myFile=paste(dataDir,"/",currYear,"/",currRates,"_",i,".txt",sep="")
          if(!file.exists(myFile)) next
          print(myFile)
          euribor=read.table(myFile,header=T,blank.lines.skip=T,as.is=F)


################################################################################
###################### Printing selected interest rates ##############################
################################################################################

        print(euribor[dimnames(euribor)[[1]][euribor[,1]==i_euribor_current[1]],1:ncol(euribor)])

        euribor_current=(euribor[dimnames(euribor)[[1]][euribor[,1]==i_euribor_current[1]],1:ncol(eurib
or)])
        }


################################################################################
################## Creating the export file with important parameters ###################
################################################################################

DATA=matrix("numeric",nrow=nrow(expMat),ncol=13)
dimnames(DATA)=list(dimnames(expMat)[[1]],c(dimnames(expMat)[[2]],"Mat_Exp","T_Mat","Ri","PV_f
actor","Abs_Diff","K0","Mid_Px","Fi","Sub_Index","Days_Exp"))
DATA[,1]=expMat[,1]
DATA[,2]=expMat[,2]
DATA[,3]=expMat[,3]





################################################################################
#################################### Load the library DATE########################
################################################################################
#####################Calculation of the 8 maturities to expiration####################
################################################################################

library(date)
n=mdy.date(currMonth,currDay,currYear)
for (e in 1:nrow(expMat))
        {
                m=mdy.date(expMat[e,"Exp_Month"],expMat[e,"Exp_Day"],expMat[e,"Exp_Year"])
                DATA[e,13]=m-n
                DATA[e,4]=(as.numeric(DATA[e,13])*Days_Sec)-DiffCloseExpiry*Hour_sec
                DATA[e,5]=as.numeric(DATA[e,4])/Year_Sec
        }
```

```
################################################################################
#################Calculation of parameters for the PV factor interpolation################
################################################################################

for (i in 1:nrow(expMat))
	{
		for (e in 1:length(Int_T_Exp))
		{
			if(Int_T_Exp[e]>as.numeric(DATA[i,5])) break
		}
		T_k1<-c(Int_T_Exp[e])
		rate_k1=euribor_current[e+1]



		for (e in 2:length(Int_T_Exp))
		{
			if(Int_T_Exp[e]>as.numeric(DATA[i,5])) break
		}
		T_k<-Int_T_Exp[e-1]
		rate_k=euribor_current[e-1+1]



################################################################################
##########Interpolation of the interest rates to find nearest rates to time to expiration#########
################################################################################

		Ri_1=rate_k*(T_k1-as.numeric(DATA[i,5]))/(T_k1-
T_k)+rate_k1*(as.numeric(DATA[i,5])-T_k)/(T_k1-T_k)
		DATA[i,6]=Ri_1[[1]]






################################################################################
#####################Calculation of the Present Value Factor#########################
################################################################################

		DATA[i,7]=exp(as.numeric(DATA[i,5])*(as.numeric(DATA[i,6])/100))
	}

################################################################################
######Calculation of the absolute minimal range between call and put, K0 and mid-price#######
################################################################################

for(e in 1:length(expList))
	{

################################################################################
#######################Subtract settlement prices and combine####################
################################################################################

		px_diff=matrix(expList[[e]][,2]-expList[[e]][,3],ncol=1)
		px_diff=cbind(expList[[e]],px_diff)
```

```
##################################################################################
############Find Fi,K0,minimum absolute difference and mid price#################
###############################################################################

		abs_min=c(min(abs(px_diff[,4])))
		min_row=px_diff[dimnames(px_diff)[[1]][abs(px_diff[,4])==abs_min],1:ncol(px_diff)]
		ifelse(dim(min_row)[1]==2,min_row<-min_row[-2,],NA)
		DATA[e,8]=min_row[1,4]
		DATA[e,9]=min_row[1,1]
		DATA[e,10]=c((min_row[1,2]+min_row[1,3])/2)

	DATA[e,11]=c(as.numeric(DATA[e,9])+as.numeric(DATA[e,7])*as.numeric(DATA[e,8]))

	}

###############################################################################
#########################Calculation of sub-indices################################
#################################################################################
Separate Data according to option type (Call and Put) and then select all OTM options######
####################a call if Kij>Fi and a put if Kij<Fi###############################
###############################################################################

for (e in 1:length(expList))
	{


	VStoxx_put=expList[[e]][dimnames(expList[[e]])[[1]][expList[[e]][,"Strike"]<as.numeric(DATA[e,"
K0"]) &
		expList[[e]][,"PPx"]>=TickFactor*TickSize],c("Strike","PPx")]

	VStoxx_call=expList[[e]][dimnames(expList[[e]])[[1]][expList[[e]][,"Strike"]>as.numeric(DATA[e,
"K0"]) &
		expList[[e]][,"CPx"]>=TickFactor*TickSize],c("Strike","CPx")]
		print(VStoxx_put)
		print(VStoxx_call)
		dimnames(VStoxx_call)[[2]]=list("Strike","PPx")
		K0_OOM=c(DATA[e,9],DATA[e,10])
		VStoxx_OOM=rbind(VStoxx_put,K0_OOM,VStoxx_call)
		print(VStoxx_OOM)

###############################################################################
################################Calculation of delta K###########################
###############################################################################

delta_K=matrix("numeric",nrow=nrow(VStoxx_OOM),ncol=1)
dimnames(delta_K)=list(dimnames(VStoxx_OOM)[[1]],"Delta_K")

for (i in 1:nrow(VStoxx_OOM))

	{

	if(as.numeric(VStoxx_OOM[i,1])==min(as.numeric(VStoxx_OOM[,1])))
		{
			delta_K[i,1]=as.numeric(VStoxx_OOM[i+1,1])-as.numeric(VStoxx_OOM[i,1])
			next
		}

	if(as.numeric(VStoxx_OOM[i,1])==max(as.numeric(VStoxx_OOM[,1])))
		{
			delta_K[i,1]=as.numeric(VStoxx_OOM[i,1])-as.numeric(VStoxx_OOM[i-1,1])
			next
		}
```

```
############################################################################
####################then continue to calculate the remaining delta K####################
############################################################################

	delta_K[i,1]<-(as.numeric(VStoxx_OOM[i+1,1])-as.numeric(VStoxx_OOM[i-1,1]))/2

	}

############################################################################
###################### Calculation of sub-index ####################################
############################################################################

	contri=(as.numeric(delta_K[,1])/as.numeric(VStoxx_OOM[,1])^2)*as.numeric(DATA[e,7])*as.nu
meric(VStoxx_OOM[,2])



		contri_A=sum(2*contri/as.numeric(DATA[e,5]))
		contri_B= c(((as.numeric(DATA[e,11])/as.numeric(DATA[e,9]))-
1)^2/as.numeric(DATA[e,5]))
		DATA[e,12]=sqrt(contri_A-contri_B)
	}



############################################################################
################################Saving the DATA file################################
############################################################################

Datei=paste("C:/Vola_Data_OESX/","V",IndexFile,"_Data",".txt",sep="")

write(t(i_euribor_current), file = Datei,ncolumns=13,append=T)
write(t(DATA), file =Datei,ncolumns=13,append=T)

############################################################################
###############Determine which maturities are closest to the fixed lifetime###############
############################################################################

N=FixedLifetime*Days_Sec
Nt=N/Year_Sec


	for (e in 1:nrow(expMat))
		{
			if(as.numeric(DATA[e,5])>Nt) break
		}
			T1_2<-as.numeric(DATA[e,5])
			d_2<-as.numeric(DATA[e,4])
			VStoxx_2<-as.numeric(DATA[e,12])

	for (e in 2:nrow(expMat))
		{
			if(as.numeric(DATA[e,5])>Nt) break
		}
			T1_1<-as.numeric(DATA[e-1,5])
			d_1<-as.numeric(DATA[e-1,4])
			VStoxx_1<-as.numeric(DATA[e-1,12])

ifelse(VStoxx_1==VStoxx_2,VStoxx_2<-as.numeric(DATA[e,12]),NA)
ifelse(T1_1==T1_2,T1_2<-as.numeric(DATA[e,5]),NA)
ifelse(d_1==d_2,d_2<-as.numeric(DATA[e,4]),NA)
```

```
if(as.numeric(DATA[1,5]<0))
	{
		T1_2<-as.numeric(DATA[3,5])
		d_2<-as.numeric(DATA[3,4])
		VStoxx_2<-as.numeric(DATA[3,12])
		T1_1<-as.numeric(DATA[2,5])
		d_1<-as.numeric(DATA[2,4])
		VStoxx_1<-as.numeric(DATA[2,12])

	}

if(as.numeric(DATA[1,5]>0) & as.numeric(DATA[1,5]<Int_T_Exp[1]))
	{
		T1_2<-as.numeric(DATA[3,5])
		d_2<-as.numeric(DATA[3,4])
		VStoxx_2<-as.numeric(DATA[3,12])
		T1_1<-as.numeric(DATA[2,5])
		d_1<-as.numeric(DATA[2,4])
		VStoxx_1<-as.numeric(DATA[2,12])

	}

################################################################################
##############################Calculation of the VStoxx############################
################################################################################

VSTOXX_index=c(currYear,currMonth,currDay,100*sqrt(((T1_1*(VStoxx_1)^2*((d_2-N)/(d_2-
d_1)))+(T1_2*(VStoxx_2)^2*((N-d_1)/(d_2-d_1))))*(Year_Sec/N)))
print(VSTOXX_index)


d=matrix(data=DATA[,12],ncol=8)
VSTOXX_index=c(VSTOXX_index,d)

Datei=paste("C:/Vola_Data_OESX/","V",IndexFile,".txt",sep="")

write(t(VSTOXX_index), file =Datei,ncolumns =12,append=T)

################################################################################
}}}
END of Code
```

Example: Input Data of Option on Index

1994	1	3	ODAX	C	1994	1	1950	332.4
1994	1	4	ODAX	C	1994	1	1950	309.3
1994	1	5	ODAX	C	1994	1	1950	279.8
1994	1	6	ODAX	C	1994	1	1950	273.5

Example: Input Data of Interest Rates

Date	1Wk	1M	2M	3M	4M	5M	6M	7M	8M	9M
30.12.1999	3.119	3.171	3.263	3.339	3.413	3.466	3.519	3.577	3.635	3.695
29.12.1999	3.115	3.18	3.266	3.345	3.417	3.466	3.517	3.577	3.635	3.692
28.12.1999	4.099	3.537	3.446	3.449	3.474	3.508	3.554	3.604	3.658	3.719
27.12.1999	4.033	3.538	3.446	3.45	3.474	3.506	3.552	3.603	3.656	3.717

Example: Output Data Matrix

4.1.1999

Year	M	D	N_i	T_i	R_i	PV	|Min|	F_i
1999	1	15	934200	0.029623	3.24784	1.000963	-7.6	5300
1999	2	19	3958200	0.125514	3.239365	1.004074	5.6	5300
1999	3	19	6377400	0.202226	3.235619	1.006565	19.5	5300
1999	6	18	14239800	0.451541	3.225025	1.014669	-3.1	5350
1999	9	17	22102200	0.700856	3.215892	1.022795	-12.2	5400
1999	12	17	29964600	0.950171	3.20856	1.030956	37.7	5400
2000	6	16	45689400	1.448801	3.146706	1.046645	-82.2	5600
2000	12	15	61414200	1.947432	3.079545	1.061807	188.1	5400

K_0	Mid Px	Sub-I	D rem.
112.4	5292.393	0.323222	11
244.7	5305.623	0.311934	46
328.45	5319.628	0.369674	74
500.25	5346.855	0.374346	165
623.3	5387.522	0.326149	256
735.95	5438.867	0.366329	347
905.7	5513.966	0.351485	529
1043.15	5599.726	0.212616	711

Example: Output Data of Main Index and its Sub-Indices

Year	Month	Day	VSMI	Sub-Index 1M	Sub-Index 2M	Sub-Index 3M	Sub-Index 1Q	Sub-Index 2Q
1999	1	4	30.07932483	0.334989457	0.293905892	0.355275198	0.344878943	0.276548482
1999	1	5	29.20032525	0.320161311	0.28738196	0.350393476	0.342081461	0.280957347
1999	1	6	30.07195368	0.326212453	0.297339729	0.345481761	0.340211701	0.279632143
1999	1	7	32.46603348	0.353678408	0.321573993	0.372846614	0.354822069	0.303902602
1999	1	8	32.71084455	0.323453422	0.327397239	0.363730987	0.353210291	0.30171236

Sub-Index 3Q	Sub-Index 1H	Sub-Index 2H
0.325626272	0.277964083	0.172440214
0.324113172	0.275296012	0.182708849
0.324633282	0.279314788	0.193071195
0.332330269	0.287869518	0.196841018

8. Mathematical Appendix

Derivation of New Methodology

To get to the theoretical formula of the new methodology one must first replicate a variance swap. A variance swap is a forward contract on realized volatility. The replication of a variance swap entails a dynamic hedging procedure of a so-called log contract. Assuming that the evolution of stock prices S can be express as:

$$\frac{dS_t}{S_t} = m(t,...)dt + s\,(t,...)dZ_t \qquad [1]$$

where,

dS_t = small change in stock price in time t.

S_t = stock price in time t.

m = drift parameter

s = volatility

T = time of maturity

and assuming the stock pays no dividends, then the theoretical definition of realized variance V for a given price history is the continuous integral of the form:

$$V = \frac{1}{T}\int_0^T s^{\,2}(t,...)dZ_t \qquad [2]$$

This is an appropriate approximation of the variance of daily returns used in the contract terms of most variance swaps. Now, the main idea behind the replication strategy is to create a position that, over a small incremental movement of time generates a payoff proportional to the adjustment in variance of the stock during that time.

Therefore by applying Ito's lemma to $\log S_t$, one derives:

$$d(\log S_t) = (m - \frac{1}{2}s^{\,2})dt + s\,dZ_t \qquad [3]$$

and then subtracting equation [3] from equation [1], one derives:

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{1}{2}s^{\,2}dt \qquad [4]$$

Note that the dependence on the drift parameter has been eliminated. Summing equation [4] over all times from 0 to T results in the continuously-sampled realized variance:

$$V \equiv \frac{1}{T}\int_0^T s^{2} dt$$

$$= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \qquad [5]$$

The identity above illustrates the replication strategy for realized variance. This captures the realized variance of the stock from inception to expiration at time T. The first term in the brackets outlines the net outcome of continuous rebalancing of a long stock position of value $\frac{1}{S_t}$ shares. One can take this risk-neutral first term to obtain the cost of replication directly. This can be expressed as:

$$\int_0^T \frac{dS_t}{S_t} = rT \qquad [6]$$

This shows that the shares position, which is continuously rebalanced, has a forward price that grows at the risk-less rate.

The second term within the brackets represents a static short position in a contract which at expiration has a payoff equivalent to the logarithm of the total return over the period 0 to maturity at time T. By duplicating this log payoff with liquid options (i.e. a combination of OTM calls for high stock values and OTM puts for low stock values), one can rewrite the log payoff of equation [5] as:

$$\log\frac{S_T}{S_0} = \log\frac{S_T}{S_*} + \log\frac{S_*}{S_0} \qquad [7]$$

whereby S_* represents the boundary between calls and puts. Keeping the second term constant in equation [7], independent of the final stock price S_T, means that only the first term of equation [7] must be replicated.

Future values of $-\log \frac{S_T}{S_*}$ can be expressed as the follows:

$$-\log \frac{S_T}{S_*} = \frac{-S_T - S_*}{S_*} \qquad \text{(forward contract)}$$

$$+\int_0^{S_*} \frac{1}{K^2} Max(K - S_T, 0)dK \qquad \text{(put options) [8]}$$

$$+\int_{S_*}^{\infty} \frac{1}{K^2} Max(S_T - K, 0)dK \qquad \text{(call options)}$$

Note that all contracts expire at time T.

Now the expected fear value of future variance implied in the OTM call and put options can be expressed theoretically as:

$$s^2 = \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \log \frac{S_T}{S_0}\right] \qquad [9]$$

Illustrating equation [9] with the identities of equations [6], [7] and [8] and setting:

$S_T = S_0 e^{rT}$ (the forward value of the stock price at maturity time T)

and

$\frac{-S_T - S_*}{S_*} = \frac{S_0}{S_*} e^{rT} - 1$ (the fair value of the forward contract)

results in the identity which is displayed below:

$$s^2 = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*} e^{rT} - 1\right) - \log\left(\frac{S_*}{S_0}\right) + e^{rT} \int_0^{S_*} \frac{1}{K^2} P(K)dK + e^{rT} \int_{S*}^{\infty} \frac{1}{K^2} C(K)dK \right) \qquad [10]$$

where P(K) and C(K), respectively denote the current fair price of a put and call option of strike K.

The terms are from left to right:

1. The financing cost of rebalancing the position in the underlying shares.

2. A short position in $\frac{1}{S_*}$ forward contracts struck at S_*.

3. A short position in a log contract paying $\log\left(\frac{S_*}{S_0}\right)$ at expiration.

4. A long position in $\frac{1}{K^2}$ put options with price P struck at K, for a continuum of all OTM strikes.

5. A long position in $\frac{1}{K^2}$ call options with price C struck at K, for a continuum of all OTM strikes.

Concentrating on the second and third terms in equation [10], which together represent the log payoff, we can transform these terms to form the identity $\frac{1}{T}\left(\frac{F}{S_*}-1\right)^2$. Now by isolating the second and third terms in equation [10], one gets:

$$= \frac{2}{T}\left(-\left(\frac{S_0}{S_*}e^{rT}-1\right)-\log\left(\frac{S_*}{S_0}\right)\right)$$

which gives, using both forward values :

$$= \frac{2}{T}\left(-\log\left(\frac{S_*}{S_0}e^{rT}\right)-\left(\frac{S_0}{S_*}e^{rT}-1\right)\right)$$

and by expressing the forward values with the denotation F :

$$= \frac{2}{T}\left(\log\left(\frac{F}{S_0}\right)-\left(\frac{F}{S_*}-1\right)\right) \qquad [11]$$

with:

$$\log\left(\frac{F}{S_0}\right) = \log\left(1+\left(\frac{F}{S_*}-1\right)\right)$$

$$\approx \left(\frac{F}{S_*}-1\right)-\frac{1}{2}\left(\frac{F}{S_*}-1\right)^2+\cdots$$

introducing the approximation in equation [11] assuming that S_* is slightly smaller than F:

$$\cong \frac{2}{T}\left(\left(\frac{F}{S_*}-1\right)-\frac{1}{2}\left(\frac{F}{S_*}-1\right)^2-\left(\frac{F}{S_*}-1\right)\right)$$

which is equivalent to:

$$= \frac{1}{T}\left(\frac{F}{S_*}-1\right)^2$$

Now by substituting K_0 for S_*, where K_0 a strike is slightly smaller than F, denotes the boundary between OTM calls and puts:

$$= \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2 \qquad [12]$$

Terms [4] and [5] of equation [10] can be simplified as follows:

$$e^{rT}\int \frac{1}{K^2} P(K)dK \rightarrow e^{rT}\sum_i \frac{1}{K_i^2} P(K_i)\Delta K_i \qquad [13]$$

$$e^{rT}\int \frac{1}{K^2} C(K)dK \rightarrow e^{rT}\sum_i \frac{1}{K_i^2} C(K_i)\Delta K_i \qquad [14]$$

Therefore a fusion of equations [12], [13] and [14] is actually an identity of equation [10]:

$$s^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} \cdot e^{rT} \cdot M(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2$$

i.e. discrete version of the theoretical formula.

Where,

1. $M(K_i)$ is the price of the OTM (either a put or a call) option of strike K_i.
2. ΔK_i is the distance between the midpoints of the strike intervals (i-1,i) and (i,i+1).
3. F is the forward and K_0 is the strike right below the forward.

Mathematical Glossary

Stochastic

Stochastic is synonymous with "random." The word is of Greek origin and means "pertaining to chance" (Parzen 1962, p. 7). It is used to indicate that a particular subject is seen from point of view of randomness. Stochastic is often used as counterpart of the word "deterministic," which means that random phenomena are not involved. Therefore, stochastic models are based on random trials, while deterministic models always produce the same output for a given starting condition.

Random Walk

A random process consisting of a sequence of discrete steps of fixed length. The random thermal perturbations in a liquid are responsible for a random walk phenomenon known as Brownian motion, and the collisions of molecules in a gas are a random walk responsible for diffusion. Random walks have interesting mathematical properties that vary greatly depending on the dimension in which the walk occurs and whether it is confined to a lattice.

Brownian Motion

The random walk motion of small particles suspended in a fluid due to bombardment by molecules obeying a Maxwellian velocity distribution (i.e., random walk with random step sizes). The phenomenon was first observed by Jan Ingenhousz in 1785, but was subsequently rediscovered by Brown in 1828. Einstein used kinetic theory to derive the diffusion constant for such motion.

Wiener Process

A continuous-time stochastic process *W*(*t*) for $t \geq 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 \leq s < t$, and increments for non-overlapping time intervals are independent. Brownian motion (i.e., random walk with random step sizes) is the most common example of a Wiener process.

Diffusion

For a continuous random walk , the number of step that must be taken by a particle to travel a specific distance.

Gaussian

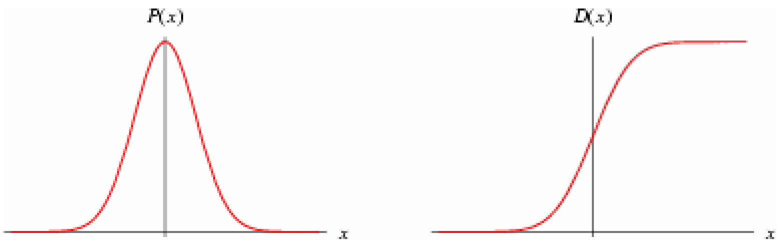
In one dimension, the Gaussian function is the probability function of the normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},$$

A normal distribution in a variate X with mean and variance is a statistic distribution with probability function.

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

While statisticians and mathematicians uniformly use the term "normal distribution" for this distribution, physicists sometimes call it a Gaussian distribution and, because of its curved flaring shape, social scientists refer to it as the "bell curve."



The quantity commonly referred to as "the" mean of a set of values is the arithmetic mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_n,$$

Variance

For a single variate X having a distribution P(x) with known population mean , the population variance , commonly also written , is defined as

$$\sigma^2 \equiv \langle (x-\mu)^2 \rangle,$$

whereas the population mean and denotes the expectation value of X. For a discrete distribution with N possible values of, the population variance is therefore

$$\sigma^2 = \sum_{i=1}^{N} P(x_i)(x_i - \mu)^2,$$

whereas for a continuous distribution, it is given by

$$\sigma^2 = \int P(x)(x-\mu)^2\, dx.$$

Discrete Distribution

A statistical distribution whose variables can take on only discrete values.

Continuous Distribution

A statistical distribution for which the variables may take on a continuous range of values.

Statistical Distribution

The distribution of a variable is a description of the relative numbers of times each possible outcome will occur in a number of trials. The function describing the distribution is called the probability function, and the function describing the cumulative probability that a given value or any value smaller than it will occur is called the distribution function.